BACHELOR OF COMMERCE

BCOM 105

BUSINESS MATHEMATICS

Directorate of Distance Education
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CONTENTS

Lesson. 1 Permutations & Combinations

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Structure:

1.0 Learning Objectives

The following are learning objective of Linear inequalities:

- To define a **permutations & combination** and explain how to calculate one.
- To identify the rule of permutations and combinations.
- To distinguish the similarities and differences between permutations and combinations.
- To correctly choose when to use permutations and combinations in order to solve problems.

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1.1 Introduction

Permutations and Combinations

Permutation and combination are all about counting and arrangements made from a certain group of data. The meaning of both these terms is explained here in this article, along with formulas and examples. This is one of the most important topics in the list of mathematics. In this lesson, will discuss all the related concepts with a diverse set of solved questions along with formulas. Moreover, practice questions based on this concept will improve your skills and help you solve any question at your own pace.

What is Permutation?

In mathematics, **permutation relates to the act of arranging all the members of a set into some sequence or order**, or if the set is already ordered, rearranging its elements, a process called permuting. Permutations occur, in more or less prominent ways, in almost every area of mathematics. They often arise when different orderings on certain finite sets are considered.

What is Combination?

The **combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter**. In smaller cases, it is possible to count the number of combinations. Combination refers to the combination of n things taken k at a time without repetition. To refer to combinations in which repetition is allowed, the terms k-selection or k-combination with repetition are often used.

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe is 472". Now we **do** care about the order. "724" won't work, nor will "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

- When order doesnot matter it is combination.
- When order does matters it is permutation.

So, we should really call this a "Permutation Lock"!

1.2 Permutation and Combinations

Definition 1.2.1 A Permutation is an ordered **Combination.**

To help you to remember, think "**P**ermutation ... **P**osition"

There are basically two types of permutation:

1. **Repetition is Allowed**: such as the lock above. It could be "333".
2. **No Repetition**: for example the first three people in a running race. You can't be first *and* second.

1. <u>Permutations with Repetition</u>

These are the easiest to calculate.

When a thing has ***n*** different types ... we have ***n*** choices each time!

For example: choosing ***3*** of those things, the permutations are:

$$\mathbf{n \times n \times n}$$

(n multiplied 3 times)

More generally: choosing ***r*** of something that has ***n*** different types, the permutations are:

n × n × ... *(r times)*

(In other words, there are **n** possibilities for the first choice, THEN there are **n** possibilites for the second choice, and so on, multplying each time.)

Which is easier to write down using an exponent of **r**:

$$n \times n \times \ldots \text{(r times)} = n^r.$$

Example 1.2.2 In the lock above, there are 10 numbers to choose from (0,1,2,3,4,5,6,7,8,9) and we choose 3 of them:

$$10 \times 10 \times \ldots \text{(3 times)} = 10^3 = 1{,}000 \text{ permutations}$$

So, the formula is simply:

$$n^r$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them, repetition is allowed, and order matters.

2. <u>Permutations without Repetition</u>

In this case, we have to **reduce** the number of available choices each time.

Example 1.2.3 what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilites, and our next choice has 15 possibilities, then 14, 13, 12, 11, ... etc. And the total permutations or way of choosing are:

$$16 \times 15 \times 14 \times 13 \times \ldots = 20{,}922{,}789{,}888{,}000$$

But ma ybe we don't want to choose them all, just 3 of them, and that is then:

$$16 \times 15 \times 14 = 3{,}360$$

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

Without repetition our choices get reduced each time.

But how do we write that mathematically? Answer: we use the "factorial function"

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$
- $1! = 1$

Note: it is generally agreed that **0! = 1**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, when we want to select **all** of the billiard balls the permutations are:

16! = 20,922,789,888,000

But when we want to select just 3 we don't want to multiply after 14. How do we do that? There is a neat trick: we divide by **13!**

$16 \times 15 \times 14 \times 13 \times 12 \ldots 13 \times 12 \ldots = 16 \times 15 \times 14$

That was neat. The 13 × 12 × ... etc gets "cancelled out", leaving only 16 × 15 × 14.

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them,

no repetitions,

order matters.

Example Our "order of 3 out of 16 pool balls example" is:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20{,}922{,}789{,}888{,}000}{6{,}227{,}020{,}800} = 3{,}360$$

(which is just the same as: 16 × 15 × 14 = 3,360)

Example 1.2.4 How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3{,}628{,}800}{40{,}320} = 90$$

(which is just the same as: 10 × 9 = 90).

Notation

Instead of writing the whole formula, people use different notations such as these:

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n - r)!}$$

Example: *P*(10, 2) = 90.

Combinations

Definition 1.2.5 The way of selecting items from a collection where order does not matter is called combination.

There are also two types of combinations (remember the order does not matter now):

1. **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
2. **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

have three scoops. How many variations OK, now we can tackle this one ...

1. Combinations with Repetition

OK, now we can tackle this one ...

let us say there are five flavors of icecream : **banana, chocolate, lemon, strawberry and vanilla**.

We can have three scoops. How many variations will there be? Let us say there are five flavors of icecream will there be?

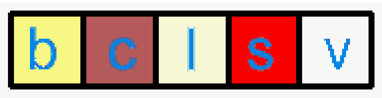
Let's use letters for the flavors: {b, c, l, s, v}. Example selections include

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)

- {b, v, v} (one of banana, two of vanilla)

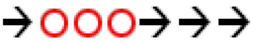
(And just to be clear: There are **n = 5** things to choose from, and we choose **r = 3** of them. Order does not matter, and we **can** repeat!)

Now, we can't describe directly that how to calculate this, but we can discuss a **special technique** that lets us work it out.



Think about the ice cream being in boxes, we could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and we will have 3 scoops of chocolate!

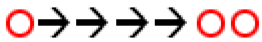
So it is like we are ordering a robot to get our ice cream, but it doesn't change anything, we still get what we want.

We can write this down as  (arrow means **move**, circle means **scoop**).

In fact the three examples above can be written like this:

{c, c, c} (3 scoops of chocolate): 

{b, l, v} (one each of banana, lemon and vanilla): 

{b, v, v} (one of banana, two of vanilla): 

OK, so instead of worrying about different flavors, we have a ***simpler*** question: "how many different ways can we arrange arrows and circles?"

Notice that there are always 3 circles (3 scoops of ice cream) and 4 arrows (we need to move 4 times to go from the 1st to 5th container).

So (being general here) there are **r + (n − 1)** positions, and we want to choose ***r*** of them to have circles.

This is like saying "we have **r + (n − 1)** pool balls and want to choose ***r*** of them". In other words it is now like the pool balls question, but with slightly changed numbers. And we can write it like this:

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!(n-1)!}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them

repetition allowed,

order doesn't matter.

Interestingly, we can look at the arrows instead of the circles, and say "we have ***r*** **+ (n − 1)** positions and want to choose **(n − 1)** of them to have arrows", and the answer is the same:

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{r!(n-1)!}$$

So, what about our example, what is the answer?

$$\frac{(3+5-1)!}{3!(5-1)!} = \frac{7!}{3!\times 4!} = \frac{5040}{6\times 24} = 35$$

There are 35 ways of having 3 scoops from five flavors of icecream.

2. Combinations without Repetition

This is how **lotteries** work. The numbers are drawn one at a time, and if we have the lucky numbers (no matter what order) we win!

The easiest way to explain it is to:

- assume that the order does matter (i.e., permutations),
- then alter it so the order does **not** matter.

Going back to our pool ball example, let's say we just want to know which 3 pool balls are chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those are the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 are chosen. These are the possibilites:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

So, the permutations have 6 times as many possibilites.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

$$3! = 3 \times 2 \times 1 = 6$$

(Another example: 4 things can be placed in $4! = 4 \times 3 \times 2 \times 1 = 24$ different ways, try it for yourself!)

So we adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in their order any more):

$$\frac{n!}{(n - r)!} \times \frac{1}{r!} = \frac{n!}{r!(n - r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them,

no repetition,

order doesn't matter.

It is often called "n choose r" (such as "16 choose 3") And is also known as the Binomial Coefficient.

Notation:

As well as the "big parentheses", people also use these notations:

$$C(n, r) = {}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Just remember the formula:

$$\frac{n!}{r!(n-r)!}$$

Example 1.2.6 Pool Balls (without order)

So, our pool ball example (now without order) is:

16! / 3!(16−3)! = *16!* / 3! × 13!

= *20,922,789,888,000* / 6 × 6,227,020,800

= 560

Or we could do it this way:

$16 \times 15 \times 14 / 3 \times 2 \times 1 = 3360 / 6 = 560$

It is interesting to also note how this formula is nice and **symmetrical**:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations.

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!.13!} = 560$$

Difference between Permutation and Combination

Permutation	Combination
Arranging people, digits, numbers, alphabets, letters, and colours	Selection of menu, food, clothes, subjects, team.
Picking a team captain, pitcher, and shortstop from a group.	Picking three team members from a group.
Picking two favourite colours, in order, from a colour brochure.	Picking two colours from a colour brochure.
Picking first, second and third place winners.	Picking three winners.

Example 1.2.7 How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that
(i) repetition of the digits is allowed?
(ii) repetition of the digits is not allowed?

Solution. Let the 3-digit number be ABC, where C is at the units place, B at the tens place and A at the hundreds place.

(i) When repetition is allowed:

The number of digits possible at C is 5. As repetition is allowed, the number of digits possible at B and A is also 5 at each.

Therefore, The total number possible 3-digit numbers =5 × 5 × 5 =125

(ii) When repetition is not allowed:

The number of digits possible at C is 5. Let's suppose one of 5 digits occupies place C, now as the repletion is not allowed, the possible digits for place B are 4 and similarly there are only 3 possible digits for place A.

Therefore, The total number of possible 3-digit numbers=5 × 4 × 3=60.

Example 1.2.8 How many 3-digits even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

Solutioin. Let the 3-digit number be ABC, where C is at the unit's place, B at the tens place and A at the hundreds place.

As the number has to even, the digits possible at C are 2 or 4 or 6. That is number of possible digits at C is 3.
Now, as the repetition is allowed, the digits possible at B is 6 (any of the 6 is okay). Similarly, at A, also, the number of digits possible is 6.
Therefore, The total number possible 3 digit numbers =6 × 6 × 3 = 108.

Example 1.2.9 How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

Solution. Let the 4 digit code be 1234.

At the first place, the number of letters possible is 10. Let's suppose any 1 of the ten occupies place 1. Now, as the repetition is not allowed, the number of letters possible at place 2 is 9. Now at 1 and 2, any

2 of the 10 alphabets have been taken. The number of alphabets left for place 3 is 8 and similarly the number of alphabets possible at 4 is 7.

The total number of 4 letter codes=10 × 9 × 8 × 7=5040.

Example 1.2.10 How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

Solution. Let the five-digit number be ABCDE. Given that first 2 digits of each number is 67. Therefore, the number is 67CDE.

As the repetition is not allowed and 6 and 7 are already taken, the digits available for place C are 0,1,2,3,4,5,8,9. The number of possible digits at place C is 8. Suppose one of them is taken at C, now the digits possible at place D is 7. And similarly, at E the possible digits are 6.

∴T he total five-digit numbers with given conditions=8 × 7 × 6=336.

Example 1.2.11 Evaluate (i) 8 ! (ii) 4 ! – 3 !

Solution. (i) 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40320

(ii) 4!-3! = (4 × 3!)-3! = 3!(4-1) = 3 × 2 × 1 × 3 = 18

Example 1.2.12 Evaluate $\frac{n!}{(n-r)!}$,when

(i) n = 6, r = 2

(ii) n = 9, r = 5

Solution. (i) Putting the value of n and r:

$$\frac{6!}{(6-2)!}=\frac{6!}{(4)!}=\frac{6\times5\times4!}{(4)!}=30$$

(ii) Putting the value of n and r:

$$\frac{9!}{(9-5)!}=\frac{9!}{(4)!}=\frac{9\times8\times7\times6\times5\times4!}{(4)!}$$
$$=9\times8\times7\times6\times5=15120$$

Example 1.2.13 How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

Solution. Total no. of digits possible for choosing = 9

No. of places for which a digit has to be taken = 3

As there is no repetition allowed;

⇒ No. of permutations = $\frac{9!}{(9-3)!}=\frac{9!}{(6)!}=\frac{9\times8\times7\times6!}{(6)!}$
$$=9\times8\times7=504$$

Example 1.2.14 How many 4-digit numbers are there with no digit repeated ?
Solution.

In questions like these we need to fill the places that are to be occupied.

To find: Four digit number (digits does not repeat)

Now we will have 4 places where 4 digits are to be put. So, At thousand's place = There are 9 ways as 0 cannot be at thousand's place = 9 ways At hundredth's place = There are 9 digits to be filled as 1 digit is already taken = 9 ways At ten's place = There are now 8 digits to be filled as 2 digits are already taken = 8 ways At unit's place = There are 7 digits that can be filled = 7 ways.

Total Number of ways to fill the four places = 9 × 9 × 8 × 7 = 4536 ways.

So a total of 4536 four digit numbers can be there with no digits repeated.

Example 1.2.15 How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

Solution. Even number means that last digit should be even, No. of possible digits at one's place = 3 (2, 4 and 6)

$\Rightarrow$ No. of permutations = ${}^{3}P_{1} = \frac{3!}{(3-1)!} = \frac{3!}{(2)!} = 3$.

One of digit is taken at one's place, Number of possible digits available = 5

$\Rightarrow$ No. of permutations = ${}^{5}P_{2} = \frac{5!}{(5-2)!} = \frac{5!}{(3)!} = \frac{5\times4\times3!}{(3)!} = 20$.

Therefore, total number of permutations =3 × 20=60.

Example 1.2.16 Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?

Solution. Total no. of digits possible for choosing =5

No. of places for which a digit has to be taken =4

As there is no repetition allowed;

$\Rightarrow$ No. of permutations = ${}^{5}P_{4} = \frac{5!}{(5-4)!} = \frac{5!}{(1)!} = \frac{5\times4\times3\times2\times1}{1} = 120$

The number will be even when 2 and 4 are at one's place.

The possibility of (2,4) at one's place = $\frac{2}{5} = 0.4$

Total number of even number = 120 × 0.4 = 48.

Example 1.2.17 From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person cannot hold more than one position?

Solution. Total no. of people in committee = 8

No. of positions to be filled = 2

⇒ No. of permutations = $^{8}_{2}P = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8\times7\times6!}{6!} = 56$

Example 1.2.18 Find n if $^{n-1}P_3 : {}^{n}P_3 = 1 : 9$.

Solution. Here $^{-1}P_3 : {}^{n}P_3 = 1 : 9$

$$\frac{^{n-1}_{3}P}{^{n}_{4}P} = \frac{\frac{(n-1)!}{((n-1)-3)!}}{\frac{n!}{(n-4)!}} = \frac{(n-1)!}{(n-4)!}.\frac{(n-4)!}{n!}$$

$$\Rightarrow \quad = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n} = \frac{1}{9}$$

⇒ n = 9.

Example 1.2.19 How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.

(i) 4 letters are used at a time, (ii) all letters are used at a time,

(iii) all letters are used but first letter is a vowel?

Solution. Total number of letters in MONDAY =6

(i) No. of letters to be used = 4

⇒ No. of permutations = $^{6}P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2!}{2!} = 360$

(ii) No. of letters to be used = 6

⇒ No. of permutations = $^{6}P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6\times5\times4\times3\times2\times1}{1} = 720$

(iii) No. of vowels in MONDAY = 2 (O and A)

⇒ No. of permutations in vowel = $^{2}P_1 = \frac{2!}{(2-1)!} = \frac{2!}{(1)!} = 2$.

Now, remaining places = 5

Remaining letters to be used = 5

$\Rightarrow$ No. of permutations = ${}^5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5\times4\times3\times2\times1}{1} = 120$

Therefore, total number of permutations = 2 × 120 =240.

Example 1.2.20 In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

Solution. Total number of letters in MISSISSIPPI =11

Letter Number of occurrence

M	1
I	4
S	4
P	2

$\Rightarrow$ Number of permutations = $\frac{11!}{1!.4!.4!.2!} = 3465.$

We take that 4 I's come together, and they are treated as 1 letter,

$\therefore$ Total number of letters=11-4+1=8

$\Rightarrow$ Number of permutations = $\frac{8!}{1!4!2!} = 840$

Therefore, total number of permutations where four I's don't come together = 34650 - 840 = 33810.

Example 1.2.21 If ${}^nC_8 = {}^nC_2$, find ${}^nC_2$.

Solution. Given: ${}^nC_8 = {}^nC_2$

We know that if $^nC_r = {}^nC_p$ then either r = p or r = n – p

Here $^nC_8 = {}^nC_2$

⇒ 8 = n – 2

⇒ n = 10

Now,

$$\therefore {}^nC_2 = {}^{10}C_2 = \frac{10!}{2!(10-2)!} \qquad \left(\therefore {}^nC_r = \frac{n!}{r!(n-r)!}\right)$$

$$= \frac{10!}{2!(8)!} = \frac{10\times9\times8!}{2\times1(8)!} = 45.$$

Example 1.2.22 How many chords can be drawn through 21 points on a circle ?

Solution. Given: 21 points on a circle

We know that we require two points on the circle to draw a chord

∴ Number of chords is:

$$\Rightarrow \quad {}^{21}C_2 = \frac{21!}{2!(21-2)!} = \frac{21.20.19!}{2!(19)!} = 210$$

∴ Total number of chords can be drawn are 210

Example 1.2.23 In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

Solution. Given: 5 boys and 4 girls are in total

We can select 3 boys from 5 boys in $^5C_3$ ways

Similarly, we can select 3 boys from 54 girls in $^4C_3$ ways

∴ No. of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3$

$$\Rightarrow {}^5C_3 \times {}^4C_3 = {}^5C_3 \times {}^4C_3 = \frac{5!}{3!(5-3)!} \times \frac{4!}{3!(4-3)!} = 10 \times 4$$

$\Rightarrow {}^5C_3 \times {}^4C_3 = 10 \times 4 = 40$

∴ No. of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3 = 40$ ways.

Example 1.2.24 Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Solution. Given: 6 red balls, 5 white balls and 5 blue balls

We can select 3 red balls from 6 red balls in ${}^6C_3$ ways

Similarly, We can select 3 white balls from 5 white balls in ${}^5C_3$ ways

Similarly, We can select 3 blue balls from 5 blue balls in ${}^5C_3$ ways

∴ No. of ways of selecting 9 balls is ${}^6C_3 \times {}^5C_3 \times {}^5C_3$

$\Rightarrow {}^6C_3 \times {}^5C_3 \times {}^5C_3 = 20 \times 10 \times 10 = 2000$.

∴ Number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour is ${}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000$.

Example 1.2.25 In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

Solution. Given: 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers

There are 5 players how can bowl, and we require 4 bowlers in a team of 11

∴ No. Of ways in which bowlers can be selected are: ${}^5C_4$

Now other players left are: 17 – 5(bowlers) = 12

Since we need 11 players in a team and already 4 bowlers are selected, we need to select 7 more players from 12.

∴ No. Of ways we can select these players are: ${}^{12}C_7$

∴ Total number of combinations possible are : ${}^{5}C_4 \times {}^{12}C_7$

$\Rightarrow {}^{5}C_4 \times {}^{12}C_7 = 5 \times 792 = 3960.$

∴ Number of ways we can select a team of 11 players where 4 players are bowlers from 17 players are: 3960'

1.3 Check Your Progress

Q.1. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

Q.2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Q.3. Determine n if

(i) ${}^{2n}C_3 : {}^{n}C_3$: 12 : 1 (ii) ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1.$

Q.4. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

1.4 Summary

- The number of permutations of n different things taken r at a time, allowing repetitions is n^r.
- The number of permutations of n different things taken all at a time is ${}^{n}P_{n} = n!$.

- The number of permutations of n things taken all at a time, in which p are alike of one kind, q are alike of second kind and r are alike of third kind and rest are different is $\frac{n!}{p!q!r!}$.
- Number of permutations of n different things taken r at a time,
  when a particular thing is to be included in each arrangement is $r.^{n-1}P_{r-1}$
  when a particular thing is always excluded, then number of arrangements $= {}^{n-1}P_r$.
- Number of permutations of n different things taken all at a time, when m specified things always come together is $m!(n+m+1)!$.
- The number of combinations of n different things taken r at a time allowing repetitions is $^{n+r-1}C_r$
- The number of ways of dividing n identical things among r persons such that each one gets at least one is $^{n-1}C_{r-1}$.
- The total number of combinations of n different objects taken r at a time in which
  (a) m particular objects are excluded = $^{n-m}C_r$
  (b) m particular objects are included = $^{n-m}C_{r-1}$.

1.5 Keywords

Factorial, Permutation is arrangement and Combination is selection.

1.6 Self-Assessment Test

Q.1. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Q.2. Find r if (i)$^{5}P_{r} = 2\,^{6}P_{r-1}$ (ii) $^{5}P_{r} = \,^{6}P_{r-1}$. Answer:- r= 3, r = 4.

Q.3. In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) words start with P and end with S,

(ii) vowels are all together,

(iii) there are always 4 letters between P and S?.

Answer:- (i) 18184400 (ii) 20160 (iii) 254016000.

Q.4. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination. Answer:- 778320.

Q.5. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

Q.6. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

1.7 Answers to check your progress

A.1. The possible outcomes after a coin toss are head and tail.

The number of possible outcomes at each coin toss is 2.

∴ The total number of possible outcomes after 3 times=$2 \times 2 \times 2$=8.

A.2. Total number of different letters in EQUATION = 8

Number of letters to be used to form a word =8

$$\Rightarrow \text{No. of permutations} = {}^{8}_{8}P = \frac{8!}{(8-8)!} = \frac{8!}{0!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{1} = 40320.$$

A.3. Given: $^{2n}C_{3} : {}^{n}C_{3}$=12 : 1

$^{2n}C_3 : {}^{n}C_3 : 12 : 1$

$$\Rightarrow \frac{^{2n}C_3}{^{n}C_3} = \frac{12}{1}$$

$$\Rightarrow \frac{\frac{2n!}{3!(2n-3)!}}{\frac{n!}{3!(n-3)!}} = \frac{12}{1}$$

$$\Rightarrow \frac{\frac{2n\,(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}}{\frac{n\,(n-1)(n-2)(n-3)!}{3!(n-3)!}} = \frac{12}{1}$$

$$\Rightarrow \frac{2n\,(2n-1)(2n-2)}{n\,(n-1)(n-2)} = \frac{12}{1}$$

$$\Rightarrow \frac{2\,(2n-1)\,2\,(n-1)}{(n-1)(n-2)} = \frac{12}{1}$$

$$\Rightarrow \frac{4\,(2n-1)}{(n-2)} = \frac{12}{1}$$

$\Rightarrow 4\times (2n—1) = 12\times(n-2)$

$\Rightarrow 8n – 4 = 12n – 24$

$\Rightarrow 12n – 8n = 24 – 4$

$\Rightarrow 4n = 20$

$\therefore n = 5$

(ii) On similar Lines.

A.4. The word DAUGHTER has 3 vowels A, E, U and 5 consonants D, G, H, T and R.

The three vowels can be chosen in $^{3}C_{2}$ as only two vowels are to be chosen.

Similarly, the five consonants can be chosen in $^5C_3$ ways.

∴ Number of choosing 2 vowels and 5 consonants would be $^3C_2 \times {}^5C_3 = 30$

∴ Total number of ways of is 30

Each of these 5 letters can be arranged in 5 ways to form different words = $^5P_5$

$$= \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120.$$

Total number of words formed would be = 30 × 120 = 3600

1.8 References/ Suggested Readings

1. Allen RG,D.: Basic Mathematics; Mcmillan, New Dehli.
2. Dowling E.T.: Mathematics for Economics; Sihahum Series, McGraw Hill. London.
3. Kapoor, V. K.: Business Mathematics: Sultan, Chan & Sons, Delhi.
4. Loomba Paul: Linear Programming: Tata McGraw Hill, New Delhi.
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Lesson. 2 Binomial Theorem

Course Name: Business Mathematics **Course Code:** BCOM 105

Semester-II **Author:** Dr Vizender Singh

Structure:

2.0 Learning Objectives

The following are learning objective of Binomial Theorem are:

- To extend polynomials and, to identify terms for a given polynomials.
- To generates row of Pascal Triangle.
- To apply in the probability theory of success or failure in a Bernoulli trial.
- To work with combinations.
- Compute binomial coefficients by formula.
- Expand powers of a binomial by Pascal's Triangle and by binomial coefficients.
- Approximate numbers using binomial expansions.

2.1 Introduction

In this lesson we will develop and prove the Binomial Theorem. This theorem tells us about the relationship between two forms of a certain type of polynomial: the factored, binomial form $(x + y)^n$, and the expanded form of this polynomial. To understand the theorem, we will first revisit a topic you have seen previously and we will discuss several other ideas needed to develop the theorem. Then we will prove the theorem using induction. Finally, we will use the theorem to expand polynomials.

2.2 Binomial Theorem

Binomial expression

An algebraic expression consisting of two terms with + *ve* or – *ve* sign between them is called a binomial expression.

For example **:** $(a+b),\ (2x-3y),\ \left(\frac{p}{x^2}-\frac{q}{x^4}\right),\ \left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

A **binomial** is a polynomial with two terms

$$\underbrace{3y^2 - 3}_{\text{Example of Binomial}}$$

What happens when we multiply a binomial by itself ... many times?

Example: a + b

a + b is a binomial (the two terms are a and b)

Let us multiply a + b by itself using Polynomial Multiplication :

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

Now take that result and multiply by a + b again:

$$(a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

And again:

$$(a^3 + 3a^2b + 3ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The calculations get longer and longer as we go, but there is some kind of pattern developing.

That pattern is summed up by the **Binomial Theorem**:

$$(a + b)^n = \sum_{k=0}^{n} {}^nC_k \, a^{n-k} b^k$$

The Binomial Theorem

Exponents of (a+b)

Now on to the binomial.

We will use the simple binomial a+b, but it could be any binomial.

Let us start with an exponent of **0** and build upwards.

Exponent of 0

When an exponent is 0, we get **1**:

$$(a + b)^0 = 1$$

Exponent of 1

When the exponent is 1, we get the original value, unchanged:

$$(a + b)^1 = a + b$$

Exponent of 2

An exponent of 2 means to multiply by itself (see how to multiply polynomials):

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

Exponent of 3

For an exponent of 3 just multiply again:

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

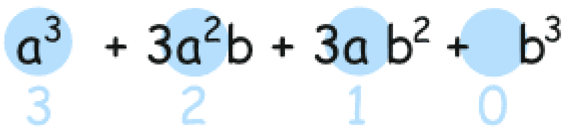
Now we talk about the pattern.

The Pattern

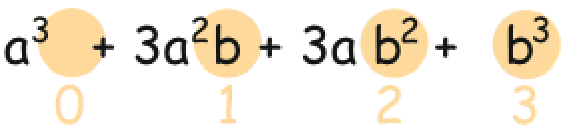
In the last result we got:

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, notice the exponents of a. They start at 3 and go down: 3, 2, 1, 0:



Likewise the exponents of b go upwards: 0, 1, 2, 3:



If we number the terms 0 to *n*, we get this:

k = 0	k = 1	k = 2	k = 3
a^3	a^2	a	1
1	b	b^2	b^3

This can be brought together into this:

$$a^{n-k} b^k$$

Consider an example to see how it works:

Example: When the exponent, *n*, is 3.

The terms are:

k = 0:	k = 1:	k = 2:	k = 3:
$a^{n-k}b^k$ $= a^{3-0}b^0$ $= a^3$	$a^{n-k}b^k$ $= a^{3-1}b^1$ $= a^2b$	$a^{n-k}b^k$ $= a^{3-2}b^2$ $= ab^2$	$a^{n-k}b^k$ $= a^{3-3}b^3$ $= b^3$

Coefficients

So far we have: $a^3 + a^2b + ab^2 + b^3$

But we **really** need: $a^3 + 3a^2b + 3ab^2 + b^3$

We are **missing the numbers** (which are called *coefficients*).

Let's look at **all the results** we got before, from $(a+b)^0$ up to $(a+b)^3$:

$$1$$
$$a + b$$
$$a^2 + 2ab + b^2$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$

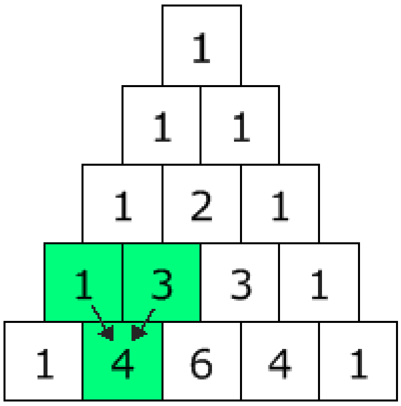
And now look at **just the coefficients** (with a "1" where a coefficient wasn't shown):

$$1$$
$$1a + 1b$$
$$1a^2 + 2ab + 1b^2$$
$$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

They actually make **Pascal's Triangle!**

Each number is just the two numbers above it added together (except for the edges, which are all "1")

(Here we have highlighted that **1+3 = 4)**



Armed with this information let us try something new ... an **exponent of 4**:

a exponents go 4,3,2,1,0: $a^4 + a^3 + a^2 + a + 1$

b exponents go 0,1,2,3,4: $a^4 + a^3b + a^2b^2 + ab^3 + b^4$

coefficients go 1,4,6,4,1: $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

And that is the correct answer (compare to the above results).

Further this pattern can be used for exponents of 5, 6, 7, ... 50, ... 112, ... you name it!

That pattern is the essence of the Binomial Theorem.

The reader can work out $(a + b)^5$ himself.

Answer (hover over): $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

As a Formula

Advancing in lesson we write it all as a formula.

We already have the exponents figured out:

$$\mathbf{a^{n-k}b^k}$$

But how do we write a formula for **"find the coefficient from Pascal's Triangle"** ... ?

Well, there **is** such a formula:

$$^{n}C_{k} = \frac{n!}{r!(n-r)!}$$

It is commonly called "n choose k" because it is how many ways to choose k elements from a set of n.

The "!" means "factorial", for example 4! = 4×3×2×1 = 24

You can read more in previous lesson entitled Combinations and Permutations.

And it matches to Pascal's Triangle like this:

(Note how the top row is row zero and also the leftmost column is zero!)

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Example 2.2.1 Row 4, term 2 in Pascal's Triangle is "6".

Let's see if the formula works:

$$^{4}C_{2} = \frac{4!}{2!(4-2)!} = \frac{4.3.2!}{2!.2!} = 6$$

Putting It All Together

The last step is to put all the terms together into one formula.

But we are adding lots of terms together ... can that be done using one formula?

This can be done using Sigma Notation allows us to sum up as many terms as we want:

Now it can all go into one formula:

$$(x + y)^{n} = \sum_{k=0}^{n} {}^{n}C_{k}\, x^{n-k} y^{k} \qquad (1)$$

The Binomial Theorem

Use It with Example.

For $n = 3$:

$$(a+b)^3 = \sum_{k=0}^{3} {}^3C_k\, a^{3-k} b^k$$

$$= {}^3C_0\, a^{3-0} b^0 + {}^3C_1\, a^{3-1} b^1 + {}^3C_2\, a^{3-2} b^2 + {}^3C_3\, a^{3-3} b^3$$

$$= 1.a^3 + 3.a^2 b^1 + 3.a\, b^2 + 1.b^3$$

$$= a^3 + a^2 b^1 + a\, b^2 + b^3$$

BUT ... it is usually **much easier** if we remember the **patterns**:

- The first term's exponents start at **n and go down**
- The second term's exponents start at **0 and go up**
- Coefficients are from Pascal's Triangle, or by calculation using $\frac{n!}{r!(n-r)!}$

Example 2.2.1 What is $(y+5)^4$

Start with exponents:	$\mathbf{y^4 5^0}$	$\mathbf{y^3 5^1}$	$\mathbf{y^2 5^2}$	$\mathbf{y^1 5^3}$	$\mathbf{y^0 5^4}$
Include Coefficients:	$\mathbf{1}y^4 5^0$	$\mathbf{4}y^3 5^1$	$\mathbf{6}y^2 5^2$	$\mathbf{4}y^1 5^3$	$\mathbf{1}y^0 5^4$

Then write down the answer (including all calculations, such as 4×5, 6×5^2, etc):

$$(y+5)^4 = y^4 + 20y^3 + 150y^2 + 500y + 625$$

The coefficient of single term can also be calculated:

Example 2.2.2 What is the coefficient for x^3 in $(2x+4)^8$.

The **exponents** for x^3 are **8 - 5** (=3) for the "2x" and **5** for the "4":

$$(2x)^3 4^5$$

(Why? Because:

2x:	8	7	6	5	4	3	2	1	0
4:	0	1	2	3	4	5	6	7	8
	$(2x)^8 4^0$	$(2x)^7 4^1$	$(2x)^6 4^2$	$(2x)^5 4^3$	$(2x)^4 4^4$	$(2x)^3 4^5$	$(2x)^2 4^6$	$(2x)^1 4^7$	$(2x)^0 4^8$

But we don't need to calculate all the other values if we only want one term.)

And let's not forget "8 choose 5" ... we can use Pascal's Triangle, or calculate directly:

$$^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8.7.6.5!}{5!.3!} = 56$$

And we get:

$$56(2x)^3 4^5$$

Which simplifies to:

$$\mathbf{458752}\ x^3$$

Some important expansions

(1) Replacing y by $-y$ in equation (1) above, we get,

$$(x - y)^n = {}^nC_0 x^{n-0} y^0 - {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 - \ldots\ldots. + (-1)^r\ {}^nC_r x^{n-r} y^r + \ldots. + (-1)^n\ {}^nC_n x^0 y^n$$

i.e., $(x - y)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r x^{n-r} y^r$

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(2) Replacing *x* by 1 and *y* by *x* in equation (i) we get,

$$(1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + ... + {}^nC_rx^r + ... + {}^nC_nx^n$$

i.e., $(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$

This is expansion of $(1+x)^n$ in ascending power of *x*.

(3) Replacing *x* by 1 and *y* by – *x* in (i) we get,

$$(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 - ..... + (-1)^r\, {}^nC_rx^r + ... + (-1)^n\, {}^nC_nx^n$$

i.e., $(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_rx^r$

(4) $(x+y)^n + (x-y)^n = 2[{}^nC_0x^ny^0 + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + ...]$ and

$$(x+y)^n - (x-y)^n = 2[{}^nC_1x^{n-1}y^1 + {}^nC_3x^{n-3}y^3 + {}^nC_5x^{n-5}y^5 + ...]$$

(5) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(6) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

General term

The general term of the expansion is $(r+1)^{th}$ term usually denoted by T_{r+1} and $T_{r+1} = {}^nC_rx^{n-r}y^r$

- In the binomial expansion of $(x-y)^n, T_{r+1} = (-1)^r\, {}^nC_rx^{n-r}y^r$
- In the binomial expansion of $(1+x)^n, T_{r+1} = {}^nC_rx^r$
- In the binomial expansion of $(1-x)^n, T_{r+1} = (-1)^r\, {}^nC_rx^r$
- In the binomial expansion of $(x+y)^n$, the p^{th} term from the end is $(n-p+2)^{th}$ term from beginning.

Independent term or Constant term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Condition : $(n-r)$ [Power of x] + r [Power of y] = 0, in the expansion of $[x+y]^n$.

Middle term

The middle term depends upon the value of n.

(1) **When *n* is even,** then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (odd). So there is only one middle term *i.e.*, $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term. $T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}x^{n/2}y^{n/2}$

(2) **When *n* is odd**, then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (even). So, there are two middle terms *i.e.*, $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ are two middle terms.

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}} \text{ and } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$$

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.

Binomial theorem for any Index

Statement: $(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ...$

$+ \frac{n(n-1)......(n-r+1)}{r!}x^r + ...\text{terms up to} \infty$

when n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If first term is not 1, then make first term unity in the following way, $(x + y)^n = x^n\left[1 + \frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

General term : $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions

i. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...+ \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + ...$

ii. $(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - .... + \frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r + ...$

iii. $(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...+ \frac{n(n+1)......(n+r-1)}{r!}x^r + ...$

iv. $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ...+ \frac{n(n+1)......(n+r-1)}{r!}(-x)^r + ...$

v. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...\infty$

vi. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...\infty$

vii. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$

viii. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...\infty$

ix. $(1 + x)^{-3} = 1 - 3x + 6x^2 - ....\infty$

x. $(1 - x)^{-3} = 1 + 3x + 6x^2 + ...\infty$

Example 2.2.3 Find the 6^{th} term in expansion of $\left(2x^2 - \frac{1}{3x^2}\right)^{10}$.

Solution. Applying $T_{r+1} = {}^nC_r x^{n-r} a^r$ for $(x+a)^n$

$$\text{Hence } T_6 = {}^{10}C_5 (2x^2)^5 \left(-\frac{1}{3x^2}\right)^5$$

$$= -\frac{10!}{5!5!} 32 \times \frac{1}{243} = -\frac{896}{27}$$

Example 2.2.4 If the ratio of the coefficient of third and fourth term in the expansion of

$\left(x - \frac{1}{2x}\right)^n$ is 1 : 2, then find the value of *n*.

Solution. $T_3 = {}^nC_2 (x)^{n-2} \left(-\frac{1}{2x}\right)^2$ and $T_4 = {}^nC_3 (x)^{n-3} \left(-\frac{1}{2x}\right)^3$

But according to the condition,

$$\frac{-n(n-1) \times 3 \times 2 \times 1 \times 8}{n(n-1)(n-2) \times 2 \times 1 \times 4} = \frac{1}{2} \Rightarrow n = -10$$

Example 2.2.5 Find the r^{th} term in the expansion of $(a + 2x)^n$.

Solution. r^{th} term of $(a+2x)^n$ is ${}^nC_{r-1}(a)^{n-r+1}(2x)^{r-1}$

$$= \frac{n!}{(n-r+1)!(r-1)!} a^{n-r+1} (2x)^{r-1}$$

$$= \frac{n(n-1).....(n-r+2)}{(r-1)!} a^{n-r+1} (2x)^{r-1}$$

Example 2.2.6 If x^4 occurs in the r^{th} term in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$, then find r.

Solution. $T_r = {}^{15}C_{r-1} (x^4)^{16-r} \left(\frac{1}{x^3}\right)^{r-1} = {}^{15}C_{r-1} x^{67-7r}$

$$\Rightarrow 67 - 7r = 4 \Rightarrow r = 9.$$

Example 2.2.7 The first 3 terms in the expansion of $(1 + ax)^n \quad (n \neq 0)$ are 1, 6x and $16x^2$. Then find the value of a and n .

Solution. $T_1 = {}^nC_0 = 1$ …..(i)

$T_2 = {}^nC_1 ax = 6x$ …..(ii)

$T_3 = {}^nC_2 (ax)^2 = 16x^2$ …..(iii)

From (ii), $\frac{n!}{(n-1)!} a = 6 \Rightarrow na = 6$ …..(iv)

From (iii), $\frac{n(n-1)}{2} a^2 = 16$ …..(v)

Using (iv) in (v), we have

$$\Rightarrow \frac{n(n-1)}{2} \left(\frac{6}{n}\right)^2 = 16$$

$$\Rightarrow \frac{n^2(1 - \frac{1}{n})}{2} \frac{36}{n^2} = 16$$

$$\Rightarrow 1 - \frac{1}{n} = \frac{8}{9}$$

$$\Rightarrow 1 - \frac{8}{9} = \frac{1}{n}$$

$$\Rightarrow n = 9$$

Using (iv), gives $a = \frac{2}{3}$.

Example 2.2.8 If *A* and *B* are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then find the relation between A and B.

Solution. Here

$$\frac{\text{Coefficient of } x^n \text{ in expansion of} (1+x)^{2n}}{\text{Coefficient of } x^n \text{ in expansion of } (1+x)^{2n-1}}$$

$$= \frac{^{2n}C_n}{^{(2n-1)}C_n} = \frac{(2n)!}{n!n!} \times \frac{(n-1)!n!}{(2n-1)!}$$

$$= \frac{(2n)(2n-1)!(n-1)!}{n(n-1)!\,(2n-1)!} = \frac{2n}{n} = 2:1$$

$$\Rightarrow \frac{A}{B} = \frac{2}{1} \Rightarrow A = 2B.$$

Example 2.2.9 In the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$, find the coefficient of x^4.

Solution. In the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$, the general term is $T_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} . \left(-\frac{3}{x^2}\right)^r$

$$= {}^{10}C_r (-1)^r . \frac{3^r}{2^{10-r}} x^{10-r-2r}$$

Here, the exponent of x is $10 - 3r = 4 \Rightarrow r = 2$

$$T_{2+1} = {}^{10}C_2 \left(\frac{x}{2}\right)^8 \left(-\frac{3}{x^2}\right)^2 = \frac{10.9}{1.2} . \frac{1}{2^8} . 3^2 . x^4$$

$$= \frac{405}{256} x^4$$

$\therefore$ The required coefficient $= \frac{405}{256}$.

Example 2.2.10 If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficient of x and x^2 are 3 and – 6 respectively, then find the value of m.

Solution. $(1 + x)^m (1 - x)^n$

$$= \left(1 + mx + \frac{m(m-1)x^2}{2!} + ...\right)\left(1 - nx + \frac{n(n-1)}{2!} x^2 - ....\right)$$

$$=1 + (m - n)x + \left[\frac{n^2 - n}{2} - mn + \frac{(m^2 - m)}{2}\right]x^2 + ...$$

Given, m – n = 3 or n = m – 3

Hence $\frac{n^2 - n}{2} - mn + \frac{m^2 - m}{2} = -6$

$$\Rightarrow \frac{(m - 3)(m - 4)}{2} - m(m - 3) + \frac{m^2 - m}{2} = -6$$

$$\Rightarrow m^2 - 7m + 12 - 2m^2 + 6m + m^2 - m + 12 = 0$$

$$\Rightarrow -2m + 24 = 0 \Rightarrow m = 12$$

Example 2.2.11 If the coefficient of 4^{th} term in the expansion of $(a + b)^n$ is 56, then find n. **Solution.**

$T_4 = T_{3+1} = {}^nC_3 \ a^{n-3}b^3$

$$\Rightarrow {}^nC_3 = 56 \Rightarrow \frac{n!}{3!\,(n - 3)!} = 56$$

$$\Rightarrow n(n - 1)(n - 2) = 56.6 \Rightarrow n(n - 1)(n - 2) = 8.7.6$$

$$\Rightarrow n = 8.$$

Example 2.2.12 In the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, find the term independent of x.

Solution. In the expansion of $\left(\frac{3x^2}{2} + \frac{1}{3x}\right)^9$, the general term is $T_{r+1} = {}^9C_r.\left(\frac{3x^2}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

For the term independent of *x*, 18 – 3r = 0 ⇒ r = 6

This gives the independent term

$$T_{6+1} = {}^9C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 = {}^9C_3.\frac{1}{6^3}$$

Example 2.2.13 If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924\,x^6$, then find the valoue of n.

Solution. Since *n* is even therefore $\left(\frac{n}{2}+1\right)^{th}$ term is middle term, hence

$$^nC_{n/2}(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2} = 924x^6$$

$$\Rightarrow\ x^{n/2} = x^6\ \Rightarrow n = 12.$$

Example 2.2.14 Find the greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$.

Solution. Let (r + 1)th term be the greatest term. Then

$$T_{r+1}=\sqrt{3}.\,^{20}C_r\left(\frac{1}{\sqrt{3}}\right)^r \text{ and } T_r=\sqrt{3}.\,^{20}C_{r-1}\left(\frac{1}{\sqrt{3}}\right)^{r-1}$$

Now $\frac{T_{r+1}}{T_r} = \frac{20-r+1}{r}\left(\frac{1}{\sqrt{3}}\right)$

$$\therefore\ T_{r+1}{}^3T_r \Rightarrow 20 - r + 1^3\sqrt{3}r$$

$$\Rightarrow 21 \geq r(\sqrt{3}+1) \Rightarrow r \leq \frac{21}{\sqrt{3}+1} \Rightarrow r \leq 7.686 \Rightarrow r = 7$$

Hence the greatest term is

$$T_8 = \sqrt{3}\,^{20}C_7\left(\frac{1}{\sqrt{3}}\right)^7 = \frac{25840}{9}$$

Example 2.2.15 Find the value of $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n$.

Solution. We have by binomial theorem

$$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... \qquad .....(i)$$

$$\left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + ... + C_r\frac{1}{x^r} + ... \qquad .....(ii)$$

Multiplying both sides and equating coefficient of x^r in $\frac{1}{x^n}(1+x)^{2n}$ or the coefficient of x^{n+r} in $(1+x)^{2n}$ we get the value of required expression

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$$

Example 2.2.16 Find the coefficient of x^3 in the expansion of $\frac{(1+3x)^2}{1-2x}$.

Solution. Here

$$\frac{(1+3x)^2}{1-2x} = (1+3x)^2(1-2x)^{-1}$$

$$= (1+3x)^2\left(1+2x+\frac{1.2}{2.1}(-2x)^2+....\right)$$

$$= (1+6x+9x^2)(1+2x+4x^2+8x^3+....)$$

Therefore coefficient of x^3 is $(8+24+18)=50$.

Example 2.2.17 If x is positive, then find the first negative term in the expansion of $(1+x)^{27/5}$.

Solution. Here $(1+x)^{27/5}$

$$T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}(x)^r$$

For first negative term $n-r+1<0$; $r>\frac{32}{5}$.

$\therefore$ First negative term is 8th term.

Example 2.2.18 If the three consecutive coefficient in the expansion of $(1+x)^n$ are 28, 56 and 70, then find the value of *n* .

Solution. Let the consecutive coefficient of $(1+x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$

By condition, ${}^nC_{r-1}:{}^nC_r:{}^nC_{r+1} = 6:33:110$

Now ${}^nC_{r-1}:{}^nC_r = 6:33$

$\Rightarrow 2n - 13r + 2 = 0$ …..(i)

and ${}^{n}C_{r} : {}^{n}C_{r+1} = 33 : 110$

$\Rightarrow 3n - 13r - 10 = 0$ …..(ii)

Solving (i) and (ii), we get n = 12 and r = 2.

Example. 2.2.19 Use Binomial theorem to find the value of $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

Solution. We have $(x + a)^n + (x - a)^n = 2\ [x^n + {}^{n}C_2 x^{n-2} a^2 + {}^{n}C_4 x^{n-4} a^4 + {}^{n}C_6 x^{n-6} a^6 + ...]$

Here, $n = 6, x = \sqrt{2}, a = 1;\ {}^{6}C_2 = 15, {}^{6}C_4 = 15, {}^{6}C_6 = 1$

$$\therefore (\sqrt{2} + 1)^6 (\sqrt{2} - 1)^6 = 2[(\sqrt{2})^6 + 15.(\sqrt{2})^4.1 + 15(\sqrt{2})^2.1 + 1.1]$$

$$= 2[8 + 15\times 4 + 15\times 2 + 1] = 198.$$

Example. 2.2.20 If $\dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{\sqrt{4 - x}}$ is approximately equal to $a + bx$ for small values of x, then find the value of (a, b).

Solution. Here $\dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{\sqrt{4 - x}} = \dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{2\left[1 - \dfrac{x}{4}\right]^{1/2}}$

$$= \frac{\left[1 + \frac{1}{2}(-3x) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2 + ...\right] + \left[1 + \frac{5}{3}(-x) + \frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2 + ...\right]}{2\left[1 + \frac{1}{2}\left(-\frac{x}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2 + ...\right]}$$

$$= \frac{\left[1 - \frac{19}{12}x + \frac{53}{144}x^2 - ....\right]}{\left[1 - \frac{x}{2} - \frac{1}{8}x^2 - ....\right]} = 1 - \frac{35}{24}x + ...$$

Neglecting higher powers of x, then

$$a + bx = 1 - \frac{35}{24}x \Rightarrow a = 1, b = -\frac{35}{24}.$$

2.3 Check Your Progress

Q.1. Find the coefficient of *x* in the expansion of $\left(x^2+\frac{a}{x}\right)^5$.

Q.2. In the expansion of $\left(x-\frac{1}{x}\right)^6$, Find the constant term.

Q.3 Find the coefficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$.

Q.4. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^2$, then find $C_0^2 + C_1^2 + C_2^2 + C_3^2 + ... + C_n^2$.

Q.5. Find the approximate value of $(7.995)^{1/3}$ correct to four decimal places.

Q.6. Prove that $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+..........=\sqrt{2}.$

Q.7. If the three consecutive coefficients in the expansion of $(1+x)^n$ are 28, 56 and 70, then find the value of n.

Q.8. The digit in the unit place of the number $(183!) + 3^{183}$.

Q.9. The coefficients of three successive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, then find the value of n .

2.4 Summary

- The number of terms in the expansion of $(x + y)^n$ are $(n + 1)$.
- In any term of expansion of $(x + y)^n$, the sum of the exponents of *x* and *y* is always constant = *n*.
- The binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ...$ equidistant from beginning and end are equal *i.e.*, ${}^nC_r = {}^nC_{n-r}$.

- $(x + y)^n =$ Sum of odd terms + Sum of even terms.
- In the expansion of $(x + y)^n, n \in N$ $\frac{T_{r+1}}{T_r} = \left(\frac{n - r + 1}{r}\right)\frac{y}{x}$.
- If the coefficients of p^{th}, q^{th} terms in the expansion of $(1 - x)^n$ are equal, then $p + q = n + 2$.
- The coefficient of x^{n-1} in the expansion of
  $(x - 1)(x - 2).......(x - n) = -\frac{n(n + 1)}{2}$.
- The coefficient of x^{n-1} in the expansion of
  $(x + 1)(x + 2).....(x + n) = \frac{n(n + 1)}{2}$
- For finding the greatest term in the expansion of $(x + y)^n$. we rewrite the expansion in this form $(x + y)^n = x^n\left[1 + \frac{y}{x}\right]^n$. Greatest term in $(x + y)^n = x^n$. Greatest term in $\left(1 + \frac{y}{x}\right)^n$.
- If n is odd, then $(x + y)^n + (x - y)^n$ and $(x + y)^n - (x - y)^n$, both have the same number of terms equal to $\left(\frac{n + 1}{2}\right)$.
- If n is even, then $(x + y)^n + (x - y)^n$ has $\left(\frac{n}{2} + 1\right)$ terms and $(x + y)^n - (x - y)^n$ has $\frac{n}{2}$ terms.
- There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.
- The number of terms in the expansion of $(x_1 + x_2 + ... + x_2)^n = {}^{n + r - 1}C_{r - 1}$.

2.5 Keywords

Factorial, Pascal Tringle, Arithmatic Progresion, Permutations and Combinations and Geometric Progression.

2.6 Self-Assessment Test

Q.1. In the expansion of $\left(\frac{a}{x}+bx\right)^{12}$, find the coefficient of $x^{-10.}$

Q.2. Find the term independent of x in the expansion of $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^{8}$.

Q.3. Evaluate $C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n$.

Q.4. If $|x| > 1$, then find $(1+x)^{-2}$.

Q.5. Write the first four terms in the expansion of $(1-x)^{3/2}$.

Q.6. Find the the coefficient of x^n in $\frac{(1+x)^2}{(1-x)^3}$.

Q.7. Find the coefficient of x^n in the expansion of $\frac{1}{(1-x)(3-x)}$.

Q.8. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then

find $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$.

Q.9. If the coefficient of the middle term in the expansion of $(1+x)^{2n+2}$ is p and the cofficients of middle terms in the expansion of $(1+x)^{2n+1}$ is q and r, then write the relation between p, q and r.

Q.10. If the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$, then find the value of ab.

Answer :- **1.** ${}^{12}C_1(a)^{11}(b)^1 = 12a^{11}b$ **2.** Thus term independent of x = 7.

3. 0 **4.** $\left[\frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{4}{x^5} + ...\right]$

5. $1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{x^3}{16}$ **6.** $2n^2 + 2n + 1$

7. $\frac{1}{2}\left[1 - \frac{1}{3}.\frac{1}{3^n}\right] = \frac{1}{2}\frac{[3^{n+1}-1]}{3^{n+1}}$ **8.** $2\frac{{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}}$ or $\frac{2a_2}{a_2 + a_3}$

9. q + r = p **10.** ab = 1.

2.7 Answers to check your progress

A.1. In the expansion of $\left(x^2 + \frac{a}{x}\right)^5$ the general term is $T_{r+1} = {}^5C_r(x^2)^{5-r}\left(\frac{a}{x}\right)^r = {}^5C_r a^r x^{10-3r}$

Here, exponent of *x* is $10 - 3r = 1 \Rightarrow r = 3$

$\therefore\ T_{2+1} = {}^5C_3 a^3 x = 10a^3.x$

Hence coefficient of x is $10a^3$.

A.2. In the expansion of $\left(x - \frac{1}{x}\right)^6$, the general term is ${}^6C_r x^{6-r}\left(-\frac{1}{x}\right)^r = {}^6C_r(-1)^r x^{6-2r}$

For term independent of x, $6 - 2r = 0 \Rightarrow r = 3$

Thus the required coefficient $= (-1)^3.{}^6C_3 = -20$.

A.3. $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$

$$= (1 + x)^{21}\left[\frac{(1 + x)^{10} - 1}{(1 + x) - 1}\right] = \frac{1}{x}[(1 + x)^{31} - (1 + x)^{21}]$$

$\therefore$ Coefficient of x^5 in the given expression

$= \text{Coefficient of } x^5 \text{ in } \left\{\frac{1}{x}[(1+x)^{31} - (1+x)^{21}]\right\}$

$= \text{Coefficient of } x^6 \text{ in } [(1+x)^{31} - (1+x)^{21}]$

$= {}^{31}C_6 - {}^{21}C_6.$

A.4. By Binomial theorem, we have

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n \qquad .....(i)$$

$$\text{and } \left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\left(\frac{1}{x}\right)^2 + ... + C_n\left(\frac{1}{x}\right)^n \qquad ....(ii)$$

If we multiply (i) and (ii), we get

$$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2$$

is the term independent of x and hence it is equal to the term independent of x in the product

$$(1+x)^n\left(1+\frac{1}{x}\right)^n \text{ or in } \frac{1}{x^n}(1+x)^{2n} \text{ or term containing } x^n \text{ in } (1+x)^{2n}.$$

Clearly the coefficient of x^n in $(1+x)^{2n}$ is T_{n+1} and equal to ${}^{2n}C_n = \frac{(2n)!}{n!\,n!}$.

A.5. $(7.995)^{1/3} = (8-0.005)^{1/3} = (8)^{1/3}\left[1 - \frac{0.005}{8}\right]^{1/3}$

$$= 2\left[1 - \frac{1}{3}\times\frac{0.005}{8} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2+1}\left(\frac{0.005}{8}\right)^2 + ...\right]$$

$$= 2\left[1 - \frac{0.005}{24} - \frac{\frac{1}{3}\times\frac{1}{3}}{1}\times\frac{(0.005)^2}{8} + ...\right]$$

$= 2(1-0.000208) = 2\times 0.999792 = 1.9995.$

A.6. Let the given series be identical with the expansion of $(1 + x)^n$ i.e., with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...;|x| < 1$.

Then, $nx = \frac{1}{4}$ and $\frac{n(n - 1)}{2}\ x^2 = \frac{1}{4}.\frac{3}{8} = \frac{3}{32}$

Solving these two equations for n and x. We get $x = -\frac{1}{2}$ and $n = -\frac{1}{2}$.

$\therefore$ Sum of the given series

$$= (1 + x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}.$$

A.7. Let the three consecutive coefficients be

$${}^nC_{r-1} = 28, {}^nC_r = 56 \text{ and } {}^nC_{r+1} = 70, \text{ so that}$$

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n - r + 1}{r} = \frac{56}{28} = 2$$

and $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n - r}{r + 1} = \frac{70}{56} = \frac{5}{4}$

This gives $n + 1 = 3r$ and $4n - 5 = 9r$

$\therefore\ \frac{4n - 5}{n + 1} = 3 \Rightarrow n = 8$.

A.8. We know that n! terminates in 0 for $n \geq 5$ and 3^{4n} terminator in 1, ($\because 3^4 = 81$)

$\therefore 3^{180} = (3^4)^{45}$ terminates in 1

Also $3^3 = 27$ terminates in 7

$\therefore\ 3^{183} = 3^{180}3^3$ terminates in 7.

$\therefore\ 183! + 3^{183}$ terminates in 7

i.e., the digit in the unit place = 7.

A.9. Let the coefficient of three consecutive terms i.e., $(r + 1)^{th}$, $(r + 2)^{th}$, $(r + 3)^{th}$ in expansion of $(1 + x)^n$ are 165,330 and 462 respectively then, coefficient of $(r + 1)^{th}$ term $= {}^nC_r = 165$

Coefficient of (r + 2)th term $= {}^nC_{r+1} = 330$ and

Coefficient of (r + 3)th term $= {}^nC_{r+2} = 462$

$$\therefore \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n - r}{r + 1} = 2$$

or $n - r = 2(r + 1)$ or $r = \frac{1}{3}(n - 2)$

and $\frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{n - r - 1}{r + 2} = \frac{231}{165}$

or $165(n - r - 1) = 231(r + 2)$ or $165n - 627 = 396\,r$

or $165n - 627 = 396 \times \frac{1}{3} \times (n - 2)$

or $165n - 627 = 132(n - 2)$ or $n = 11$.

2.8 References/ Suggested Readings

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Lesson. 3 Linear Inequalities

Course Name: Business Mathematics **Course Code:** BCOM 105

Semester-II **Author:** Dr Vizender Singh

Structure:

3.0 Learning Objectives

The following are learning objective of Linear inequalities:

- Identify and check solutions to inequalities with two variables.
- Able to draw the graph of each Mathematical statement.
- Graph Solution sets of linear inequalities with two variables.
- **Inequality** tells us about the **relative size** of two values.
- Represent linear inequalities as regions on the coordinate plane.
- Determine if a given point is a solution of a linear inequality.

3.1 Introduction

Linear Inequations: Mathematical expressions help us convert problem statements into entities and thus, help solve them. If the expression equates two expressions or values, then it is called an equation. For e.g. 3x + 5y = 8. On the other hand, if an expression relates two expressions or values with a ‘<’ (less than) sign, ‘>’ (greater than) sign, ‘≤’ (less than or equal) sign or ‘≥’ (greater than or equal) sign, then it is called as an Inequality.

An inequality which involves a linear function is a linear inequality. It looks like a linear equation, except that the ‘=’ sign is replaced by an inequality sign, called linear inequations. In this lesson, we will look at linear inequalities with one or two variables.

3.2 Linear Inequalities

Linear Equations in Two Variables

Definition 3.2.1 An equation that can be put in the form **ax + by + c = 0**, where a, b and c are real numbers and a, b not equal to zero is called a linear equation in two variables namely x and y. The solution for such an equation is a pair of values, one for x and one for y which further makes the two sides of an equation equal.

Solutions to Inequalities with Two Variables

The solution of a linear inequality in two variables like Ax + By > C is an ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality.

Example 3.2.2 Is (1, 2) a solution to the inequality

$$2x + 3y > 1?$$

Solution. Putting x = 1 and y = 2, in the given inequality, we have

$$2 \cdot 1 + 3 \cdot 2 > 1$$

$$2 + 5 > 1$$

$$7 > 1$$

which is true, hence the given point is solution of inequality.

How to Graph a Linear Inequality

First, graph the "equals" line, then shade in the correct area.

There are three steps:

- Rearrange the equation so "y" is on the left and everything else on the right.
- Plot the "**y =** " line (make it a solid line for **y ≤** or **y ≥**, and a dashed line for **y <** or **y**>)
- Shade above the line for a "greater than" (**y** > or **y ≥**) or below the line for a "less than" (**y <** or **y** ≤).

The graph of an inequality in two variables is the set of points that represents all solutions to the inequality. A linear inequality divides the coordinate plane into two halves by a boundary line where one half represents the solutions of the inequality. The boundary line is dashed for > and < and solid for ≤ and ≥. The half-plane that is a solution to the inequality is usually shaded.

Example 3.2.3 Graph the inequality

$$y \geq -x + 1.$$

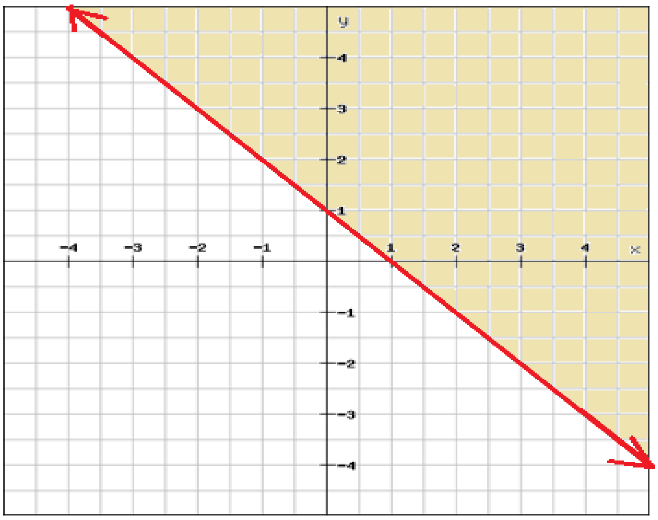
Solution. The given inequality is

$$y \geq -x + 1.$$

To plot the graph changing inequality in to equality

$$y = -x + 1$$

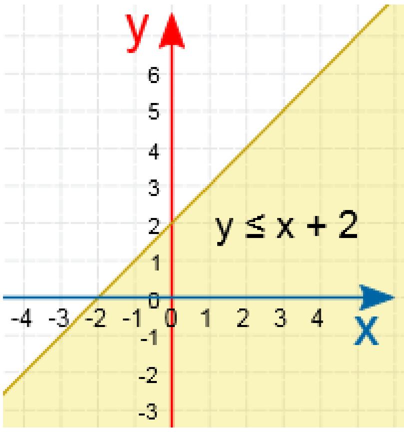
x	1	2	3	0	-1	-2	-3
y	0	-1	-2	1	2	3	4



(Graph)

Example 3.2.4 Graph the inequality y ≤ x + 2.

Solution. Writing the table of graph on similar lines as in previous example, the required graph is

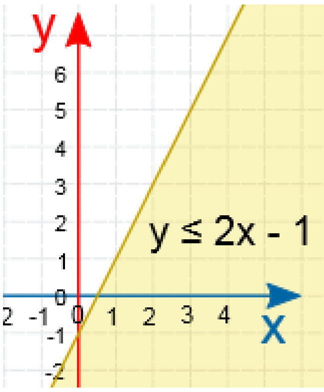


One can see the y = x + 2 line, and the shaded area is where y is less than or equal to x + 2

Example 3.2.5 Graph the inequality y ≤ 2x – 1.

Solution. The inequality already has "y" on the left and everything else on the right, so no need to rearrange.

Plot **y = 2x - 1** (as a solid line because y ≤ includes **equal to**)



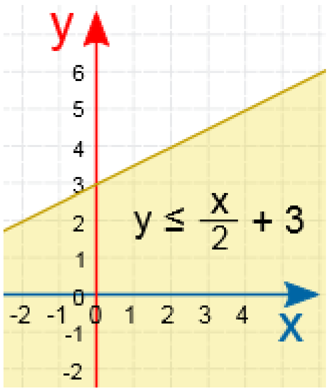
The Shaded area is below (because y is **less than** or equal to)

Example 3.2.6 Graph the inequality $2y - x \leq 6$.

Solution. We will need to rearrange this one so "y" is on its own on the left:

Start with:	$2y - x \leq 6$
Add x to both sides:	$2y \leq x + 6$
Divide all by 2:	$y \leq x/2 + 3$

Now plot **y = x/2 + 3** (as a solid line because $y \leq$ includes **equal to**)



The Shaded area is below (because y is **less than** or equal to)

Example 3.2.7 Graph the inequality y/2 + 2 > x.

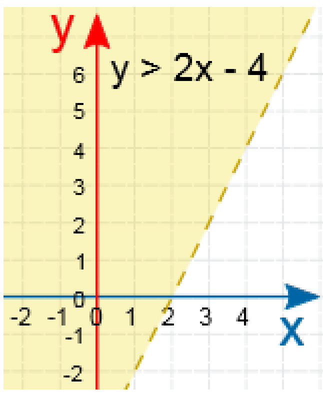
Solution. We will need to rearrange this one so "y" is on its own on the left:

Start with: y/2 + 2 > x

Subtract 2 from both sides: y/2 > x − 2

Multiply all by 2: y > 2x − 4

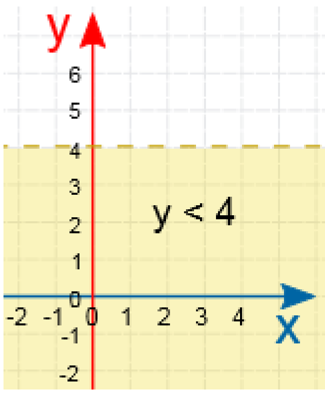
Now plot **y = 2x − 4** (as a dashed line because y> does not include equals to)



The dashed line shows that the inequality does **not** include the line **y = 2x - 4**.

Example 3.2.8 Graph the inequality y < 4 and x ≥ 1.

Solution. For inequality y < 4 the graph is

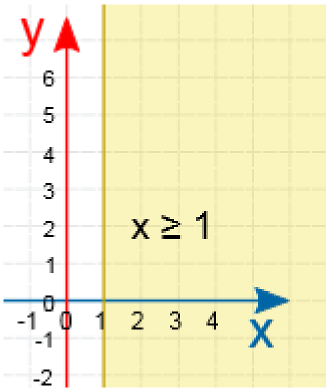


This shows where y is less than 4

(from, but not including, the line **y = 4** on down)

Notice that we have a dashed line to show that it does not include where y = 4.

For inequality $x \geq 1$the graph is

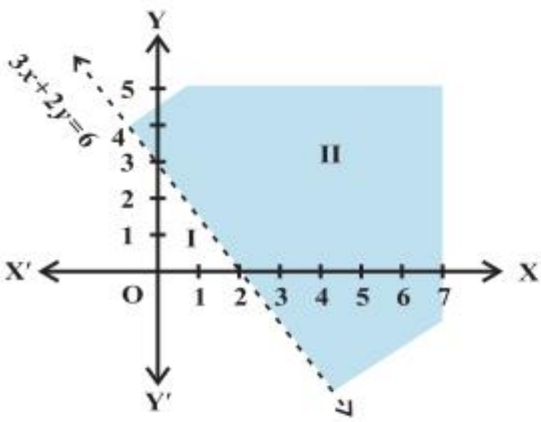


This one doesn't even have y in it!

It has the line **x=1**, and is shaded for all values of x greater than (or equal to) 1

Example 3.2.9 Solve $3x + 2y > 6$ graphically in a two-dimensional plane.

Solution. To solve the inequality, let's plot a graph of the equation $3x + 2y = 6$ as shown below:



This line divides the plane into half-planes I and II. Next, we select a point (0, 0) and determine if it satisfies the given inequality. Hence, we have

$3x + 2y > 6$

$\Rightarrow 3(0) + 2(0) > 6$

$\Rightarrow 0 > 6$ which is FALSE.

Since (0, 0) lies in the half-plane I and it does not satisfy the inequality, half-plane I, is not the solution. Also, the inequality given is a 'Strict inequality'. Hence, any point on the line represented by $3x + 2y = 0$ does not satisfy the inequality either. Therefore, the solution of the inequality is the shaded region in the diagram above.

Example 3.2.10 Solve $3x + 2y > 6$ graphically in a two-dimensional plane.

Solution. Given in equality is : $x + y < 5$

Consider: $x + y = 5$

x	0	5
y	5	0

Now draw a dotted line $x + y = 5$ in the graph ($\because$ $x + y = 5$ is excluded in the given question)

Now Consider $x + y < 5$

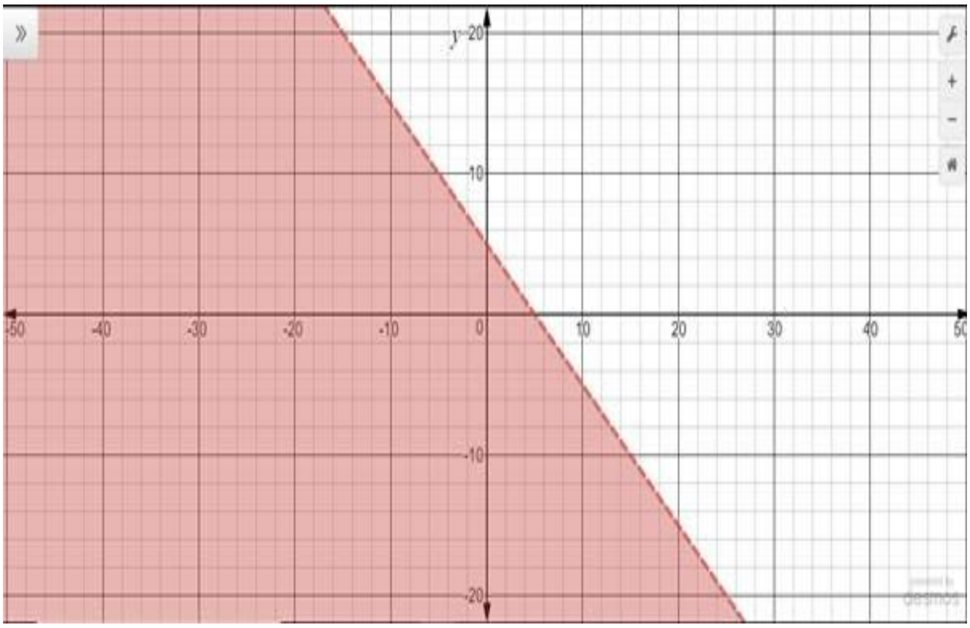
Select a point (0,0)

$\Rightarrow 0 + 0 < 5$

$\Rightarrow 0 < 5$ (this is true)

$\therefore$ Solution region of the given inequality is below the line $x + y = 5$. (That is origin is included in the region)

The graph is as follows:



Example 3.2.11 Solve the following inequalities graphically in two-dimensional plane: 2x + y ≥ 6.

Solution. Given inequality is: 2x + y ≥ 6

Consider: 2x + y = 6

x	0	3
y	6	0

Now draw a solid line 2x + y = 6 in the graph (∵2x + y = 6 is included in the given question)

Now Consider 2x + y ≥ 6

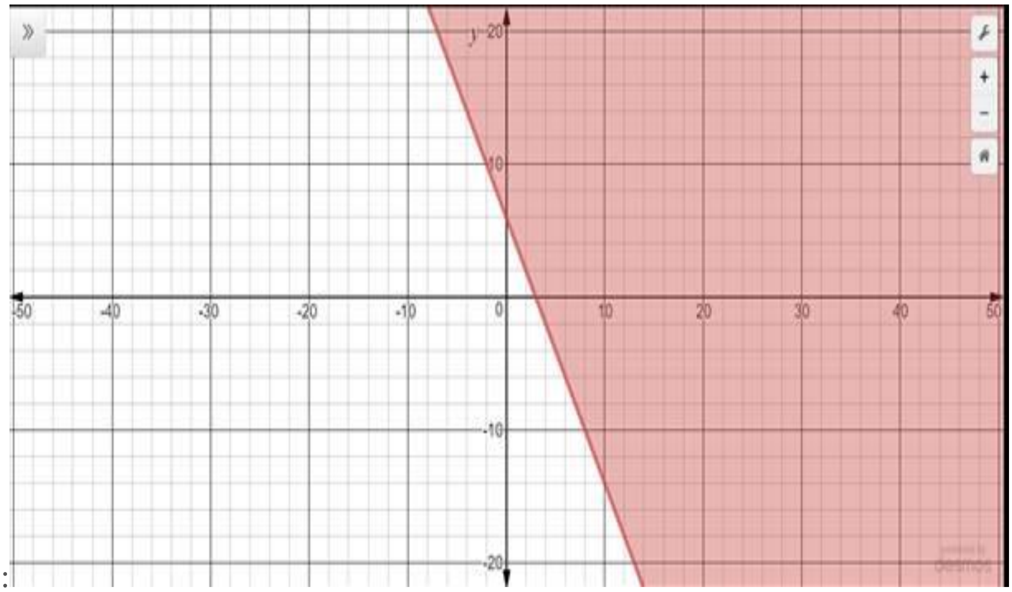
Select a point (0,0)

⇒ 2 × (0) + 0 ≥ 6

⇒ 0 ≥ 5 (this is false)

∴ Solution region of the given inequality is above the line 2x + y = 6. (Away from the origin)

The graph is as follows:



Example 3.2.12 Solve the following inequalities graphically in two-dimensional plane: $3x + 4y \le 12$.

Solution. Given: $3x + 4y \le 12$

Consider: $3x + 4y = 12$

x	0	4
y	3	0

Now draw a solid line $3x + 4y = 12$ in the graph ($\because 3x + 4y = 12$ is included in the given question)

Now Consider $3x + 4y \le 12$

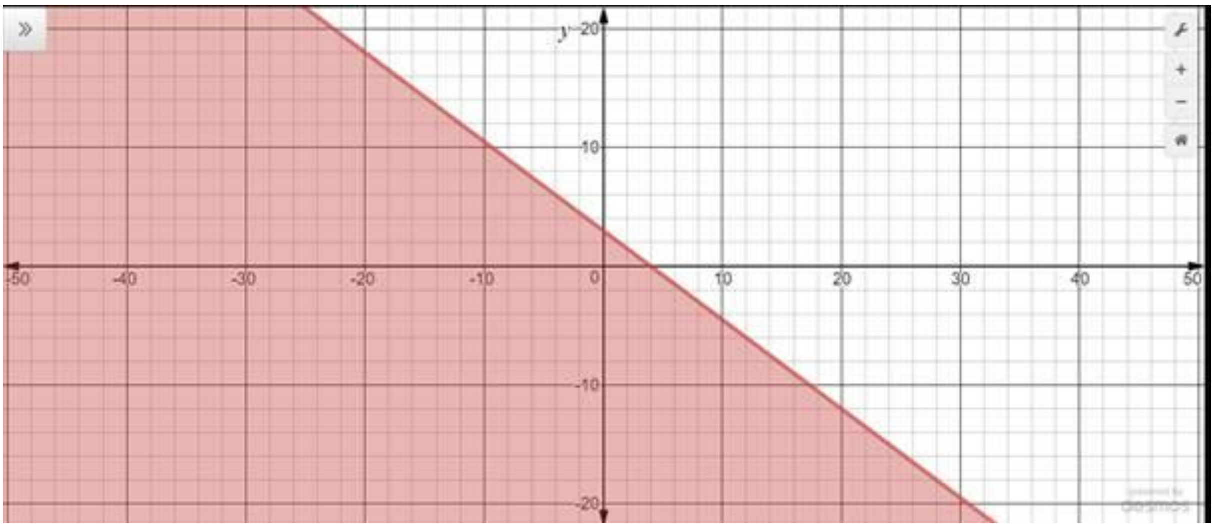
Select a point (0,0)

$\Rightarrow 3 \times (0) + 4 \times (0) \le 12$

$\Rightarrow 0 \le 12$ (this is true)

∴ Solution region of the given inequality is below the line 3x + 4y = 12. (That is origin is included in the region)

The graph is as follows:



Example 3.2.13 Solve the following inequalities graphically in two-dimensional plane: $y + 8 \geq 2x$.

Solution. Given: $y + 8 \geq 2x$

Consider: $y + 8 = 2x$

x	0	4
y	-8	0

Now draw a solid line $y + 8 = 2x$ in the graph (∵ $y + 8 = 2x$ is included in the given question)

Now Consider $y + 8 \geq 2x$

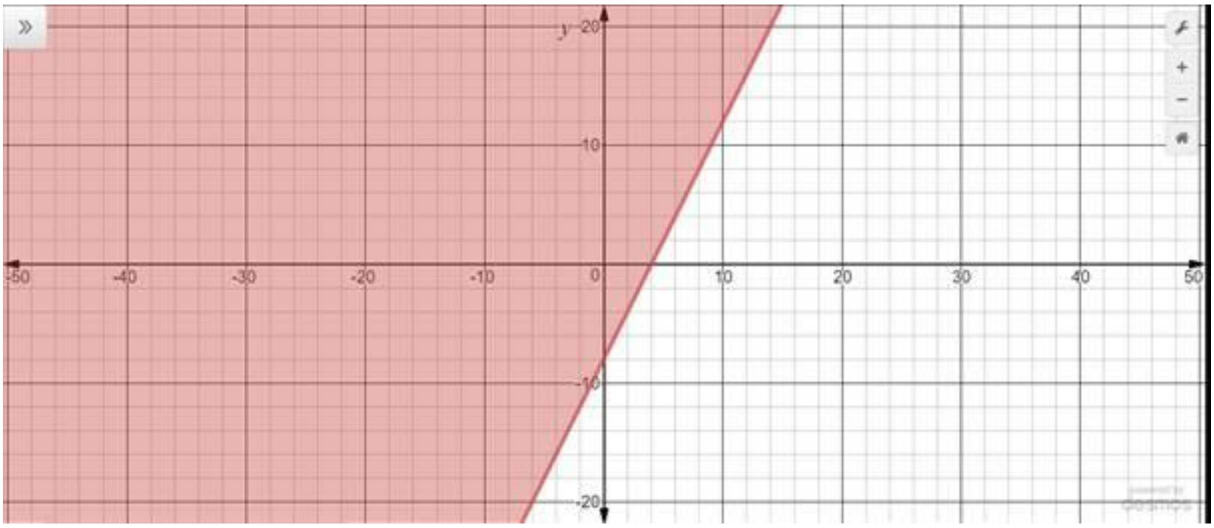
Select a point (0,0)

$\Rightarrow (0) + 8 \geq 2 \times (0)$

$\Rightarrow 0 \leq 8$ (this is true)

$\therefore$ Solution region of the given inequality is above the line y + 8 = 2x. (That is origin is included in the region)

The graph is as follows:



Example 3.2.14 Solve the following system of linear inequalities in two variables graphically.

$$x + y \geq 5$$

$$x - y \leq 3$$

Solution. To begin with, let's draw a graph of the equation x + y = 5. Now, we determine if the point (0, 0), which is lying in the half-plane I, satisfies the inequality 1. We have,

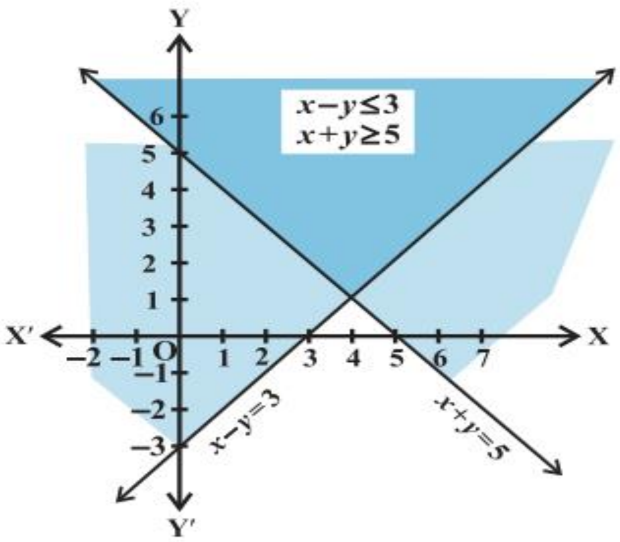
$$x + y \geq 5$$
$$\Rightarrow 0 + 0 \geq 5$$

Or, 0 ≥ 5, which is FALSE. Also, being a ‘Slack inequality, the points on the line represented by x + y = 5 satisfy the inequality 1. Hence, the solution lies in the half-plane II and includes the line.

Next, let’s draw a graph of the equation x – y = 3 on the same set of axes. Now, we determine if the point (0, 0), which is lying in the half-plane II, satisfies the inequality 2. We have,

$x - y \leq 3$
$\Rightarrow 0 - 0 \leq 3$

Or, 0 ≤ 3 which is TRUE. Also, being a ‘Slack inequality, the points on the line represented by x – y = 3, satisfy the inequality 2. Hence, the solution lies in the half-plane II and includes the line. Look at the diagram below:



The solution of the system of linear inequalities in two variables x + y ≥ 5 and x – y ≤ 3 is the region common to the two shaded regions as shown above.

Example 3.2.15 Solve the following system of inequalities graphically: x ≥ 3, y ≥ 2.

Solution. Given x ≥ 3………1

y ≥ 2……………2

Since x ≥ 3 means for any value of y the equation will be unaffected so similarly for y ≥ 2, for any value of x the equation will be unaffected.

Now putting x = 0 in the 1

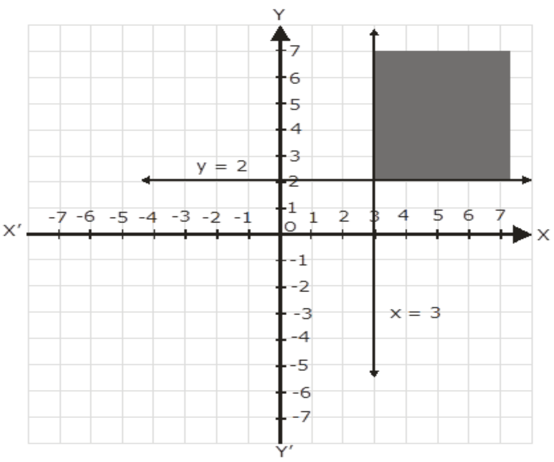
0 ≥ 3 which is not true

Putting y = 0 in 2

0 ≥ 2 which is not true again

This implies the origin doesn't satisfy in the given inequalities. The region to be included will be on the right side of the two equalities drawn on the graphs.

The shaded region is the desired region.



inequality.

⇒ 0 ≥ 2 which is not true, hence origin is not included in the solution of the

which is covered by all the given three inequalities at the same time satisfying all the given inequality.

The region to be included in the solution would be towards the left of the equality y≥ 2

The shaded region in the graph will give the answer to the required inequalities as it is the region conditions.

Example 3.2.16 Solve the following system of inequalities graphically: $3x + 2y \leq 12$, $x \geq 1$, $y \geq 2$.

Solution. Given $3x+ 2y \leq 12$

Solving for the value of x and y by putting $x = 0$ and $y = 0$ one by one

We get

$y = 6$ and $x = 4$

So the points are (0,6) and (4,0)

Now checking for (0,0)

$0 \leq 12$ which is also true,

Hence the origin lies in the plane and the required area is toward the left of the equation.

Now checking for $x \geq 1$,

The value of x would be unaffected by any value of y

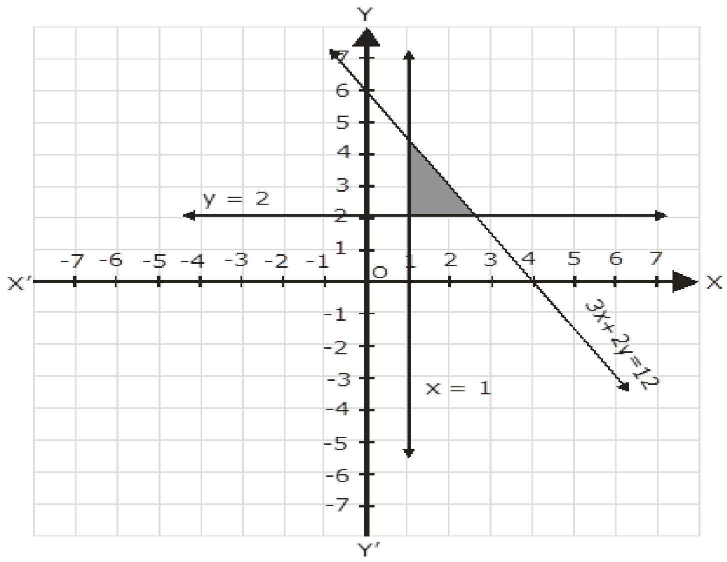
The origin would not lie on the plane

$\Rightarrow 0 \geq 1$ which is not true

The required area to be included would be on the left of the graph $x \geq 1$

Similarly, for $y \geq 2$

Value of y will be unaffected by any value of x in the given equality. Also, the origin doesn’t satisfy the given



Example 3.2.17 Solve the following system of inequalities graphically: $2x + y \geq 6$,
$3x + 4y \leq 12$.

Solution. Given $2x + y \geq 6$……………1

$3x + 4y \leq 12$ …………….2

In (1) $2x + y \geq 6$

Putting value of $x = 0$ and $y = 0$ in equation one by one, we get value of

$y = 6$ and $x = 3$

So the point for the (0,6) and (3,0)

Now checking for (0,0)

$0 \geq 6$ which is not true, hence the origin does not lies in the solution of the equality. The required region is on the right side of the graph.

Checking for $3x + 4y \leq 12$

Putting value of $x= 0$ and $y = 0$ one by one in equation

We get $y = 3$, $x = 4$

The points are (0, 3) , (4, 0)

Now checking for origin (0, 0)

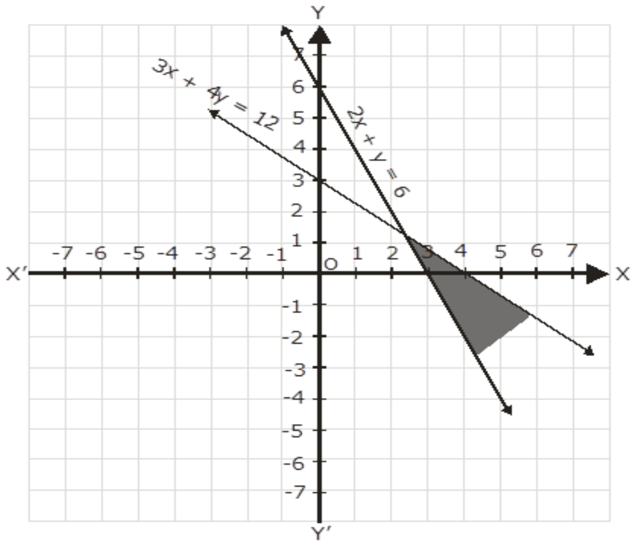
$0 \leq 12$ which is true,

so the origin lies in solution of the equation.

The region on the right of the equation is the region required.

The solution is the region which is common to the graphs of both the inequalities.

The shaded region is the required region.



Example 3.2.18 Solve the following system of inequalities graphically: $x + y \geq 4$, $2x – y < 0$.

Solution. Solving for $x + y \geq 4$

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 4 and x = 4

The points for the line are (0, 4) and (4, 0)

Checking for the origin (0, 0)

$0 \geq 4$

This is not true,

So the origin would not lie in the solution area. The required region would be on the right of line`s graph.

$2x - y < 0$

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 0 and x = 0

Putting x = 1 we get y = 2

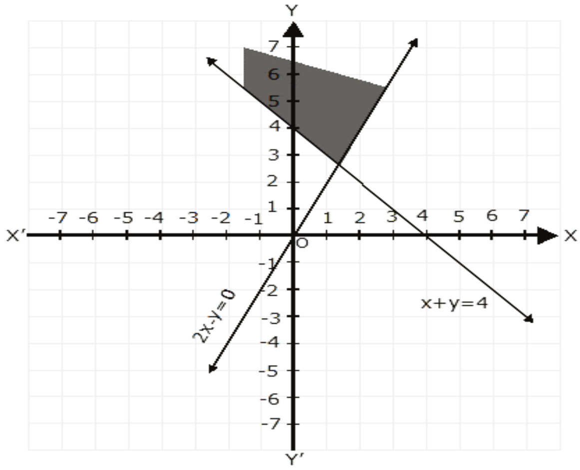
So the points for the given inequality are (0, 0) and (1,2)

Now that the origin lies on the given equation we will check for (4,0) point to check which side of the line`s graph will be included in the solution.

$\Rightarrow 8 < 0$ which is not true, hence the required region would be on the left side of the

line $2x-y < 0$

The shaded region is the required solution of the inequalities.



Example 3.2.19 Solve the following system of inequalities graphically: 2x – y >1,

x – 2y < – 1.

Solution. Given 2x – y >1………………1

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = -1 and x = 1/2 = 0.5

The points are (0,-1) and (0.5 , 0)

Checking for the origin, putting (0,0)

0 >1 , which is false

Hence the origin does not lie in the solution region. The required region would be on the right of the line`s graph.

x – 2y < – 1…………2

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 1/2 = 0.5 and x = -1

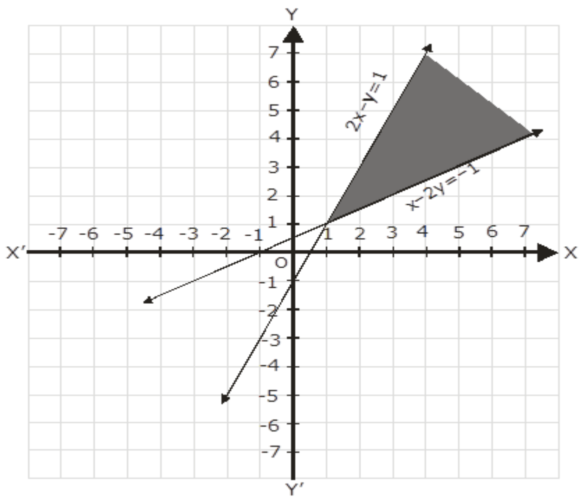
The required points are (0, 0.5) and (-1, 0)

Now checking for the origin, (0,0)

0 < -1 which is false

Hence the origin does not lies in the solution area, the required area would be on the left side of the line`s graph.

∴ the shaded area is the required solution of the given inequalities.



3.3 Check Your Progress

Q.1. Check for the points (1, 3) and (3,-2) if these ordered pairs are in the solution set of the inequality $y \geq 2x - 1$ or not.

Q.2. Solve graphically $y < 2x + 1$.

Q.3. Solve the following inequalities graphically in two-dimensional plane:

$x - y \leq 2$.

Q.4. Solve the following inequalities graphically in two-dimensional plane:

$2x - 3y > 6$.

Q.5. Solve the following inequalities graphically in two-dimensional plane:

$-3x + 2y \geq -6$.

Q.6. Draw the graph of following system of inequalities: $x + y \leq 6$, $x + y \geq 4$.

Q.7. Draw the graph of following system of inequalities: $2x + y \geq 8$, $x + 2y \geq 10$.

Q.8. Draw the graph of following system of inequalities: $x + y \leq 9$, $y > x$, $x \geq 0$.

Q.9 Draw the graph of following system of inequalities: $5x + 4y \leq 20$, $x \geq 1$, $y \geq 2$.

Q.10 Solve the following system of inequalities graphically: $3x + 4y \leq 60$, $x + 3y \leq 30$,

$x \geq 0$ $y \geq 0$

3.4 Summary

Linear inequalities with two variables have infinitely many ordered pair solutions, which can be graphed by shading in the appropriate half of a rectangular coordinate plane.

To graph the solution set of an inequality with two variables, first graph the boundary with a dashed or solid line depending on the inequality. If given a strict inequality, use a dashed line for the boundary. If given an inclusive inequality, use a solid line. Next, choose a test point not on the boundary. If the test point solves the inequality, then shade the region that contains it; otherwise, shade the opposite side.

Check your answer by testing points in and out of the shading region to verify that they solve the inequality or not.

3.5 Keywords

Linear equation in two variables; Graph Plotting; Coordinate Ponts.

3.6 Self-Assessment Test

Q.1. Is the order pair solution of given inequality?

(i) $5x - y > -2$; (-3, -5) (ii) $4x - y < -8$; (-3, -10)

(ii) $6x - 15y > -1$; (1/2, -1/3) (iv) $x - 2y \geq 2$; (2/3, -5/6).

Answers. (i) No (ii) No (iii) No (iv) No.

Q.2. Find the graphical solution of linear inequality.

(i) $y \geq -2/3\,x + 3$ (ii) $2x + 3y \leq 18$ (ii) $6x - 5y > 30$

(iv) $4x - 4y < 0$ (v) $x + > 0$ (vi) $y \leq -2$

(vii) $x < -2$ (viii) $5x \leq -4y - 12$ (ix) $4y + 2 < 3x$

(x) $5 \geq 3x - 15y$.

Q.3. Solve the following inequalities graphically in two-dimensional plane:

$y - 5x < 30$

Q.4. Solve the following inequalities graphically in two-dimensional plane:

$y < -2$

Q.5. Solve the following inequalities graphically in two-dimensional plane:

$x > -3$

Q.6. Solve the following system of inequalities graphically: $2x + y \geq 4$, $x + y \leq 3$, $2x - 3y \leq 6$

Q.7. Solve the following system of inequalities graphically: $x - 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 1$.

Q.8. Solve the following system of inequalities graphically: $4x + 3y \leq 60$, $y \geq 2x$, $x \geq 3$, $x, y \geq 0$.

Q.9. Solve the following system of inequalities graphically: $3x + 2y \leq 150$, $x + 4y \leq 80$, $x \leq 15$, $y \geq 0$, $x \geq 0$

Q.10. Solve the following system of inequalities graphically: $x + 2y \leq 10$, $x + y \geq 1$,

$x - y \leq 0$, $x \geq 0$, $y \geq 0$.

3.7 Answers to check your progress

A.1. The given inequality is; $y \geq 2x-1$

1. Let check for (1, 3) first.

Substitute x = 1 and y = 3

$3 \geq 2(1) - 1$

$3 \geq 1$

This is a true statement, so this ordered pair comes in the solution region.

2. Let check for (3,-2) now

Substitute x=3 and y = -2

$-2 \geq 2(3) - 1$

$-2 \geq 5$

This is a false statement, so this ordered pair is not comes in the solution region.

If you know the solution region then you can check it by plotting the point on the graph.

A.2. First we will find the solution of the equation by assuming x = 0 first and then x = 1.y = 2x +1

Let x = 0

y = 2(0) +1

y = 1

Let x = 1

y = 2(1) +1

y = 3

So the coordinates of the line are (0, 1) and (1, 3)

2. Joining these two points we will get the line of equation y = 2x + 1.

3. As the sign of inequality is < that is, the strict inequality so the boundary line will be a dashed line.

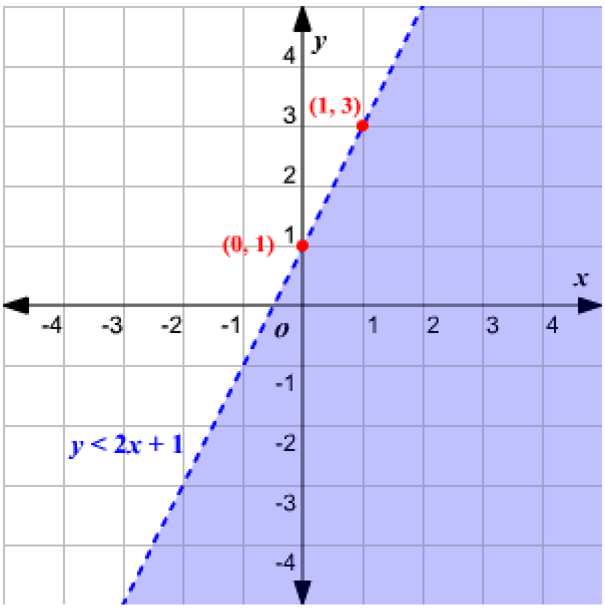
4. Now we will check for the solution region by substituting x = 0 and y = 0.

y < 2x +1

0 < 2(0) +1

This is a true statement so the solution region will be on the half-plane which contains (0, 0) coordinates.

5. So we will shade the lower half plane that is, below the boundary line.



A.3. Given: x – y ≤ 2

Consider: x – y = 2

x	0	2
y	-2	0

Now draw a solid line x – y = 2 in the graph (∵ x – y = 2 is included in the given question)

Now Consider x – y ≤ 2

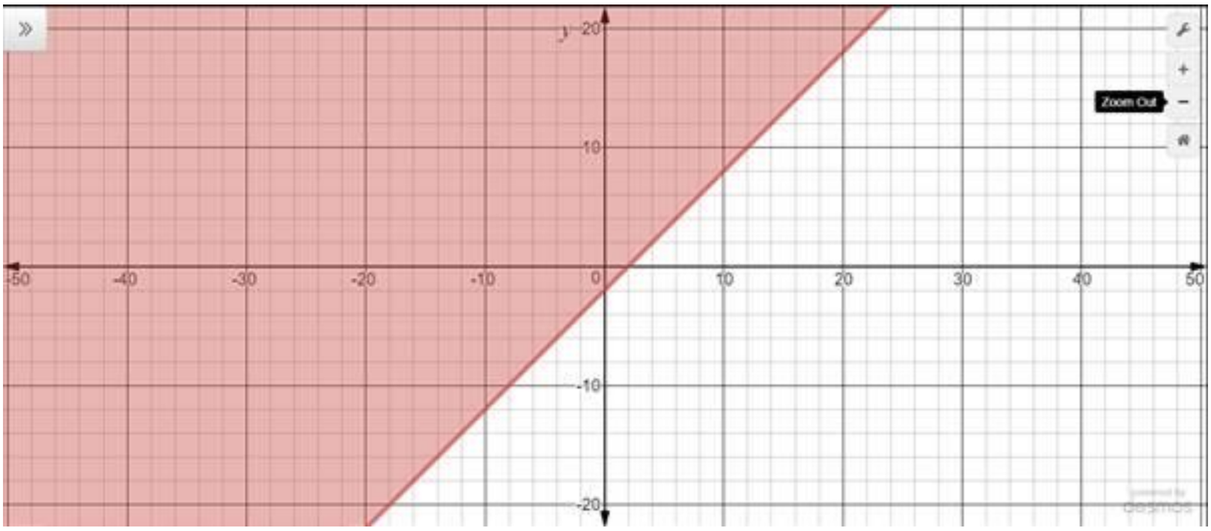
Select a point (0,0)

⇒ (0) – (0) ≤ 2

⇒ 0 ≤ 12 (this is true)

∴ Solution region of the given inequality is above the line x – y = 2. (That is origin is included in the region)

The graph is as follows:



A.4. Given: 2x – 3y > 6

Consider: 2x – 3y = 6

x	0	3
y	-2	0

Now draw a dotted line 2x – 3y = 6 in the graph (∵2x – 3y = 6 is excluded in the given question)

Now Consider 2x – 3y > 6

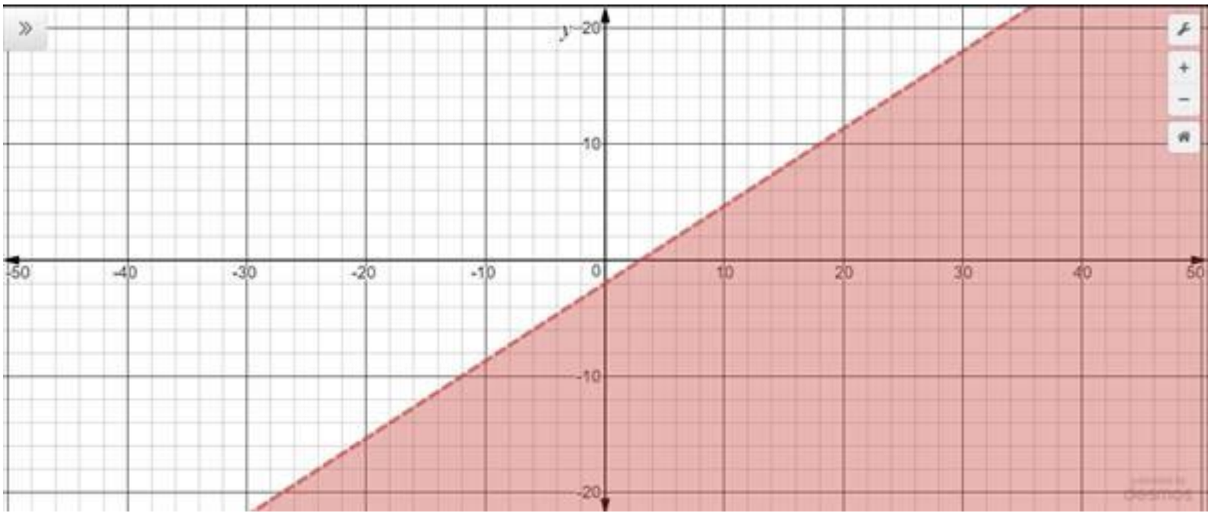
Select a point (0,0)

⇒ 2 × (0) – 3 × (0) > 6

⇒ 0>5 (this is false)

∴ Solution region of the given inequality is below the line 2x – 3y > 6. (Away from the origin)

The graph is as follows:



A.5. Given: – 3x + 2y ≥ – 6

Consider: – 3x + 2y = – 6

x	0	2
y	-3	0

Now draw a solid line – 3x + 2y = – 6 in the graph (∵– 3x + 2y = – 6 is included in the given question)

Now Consider – 3x + 2y ≥ – 6

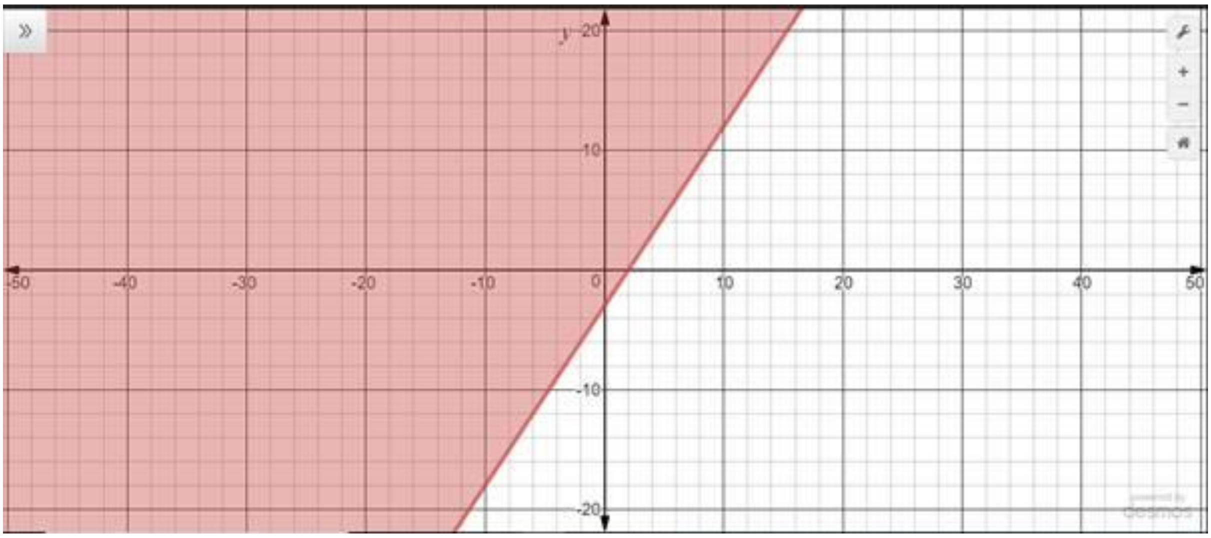
Select a point (0,0)

⇒ - 3 × (0) + 2 × (0) ≥ – 6

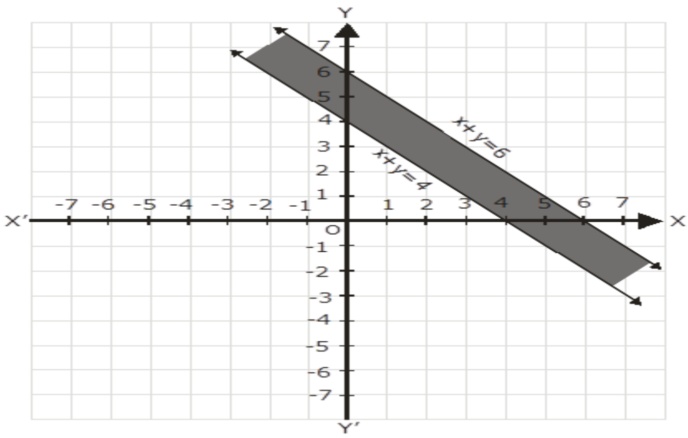
⇒ 0 ≥ – 6 (this is true)

∴ Solution region of the given inequality is above the line – 3x + 2y ≥ – 6. (That is origin is included in the region)

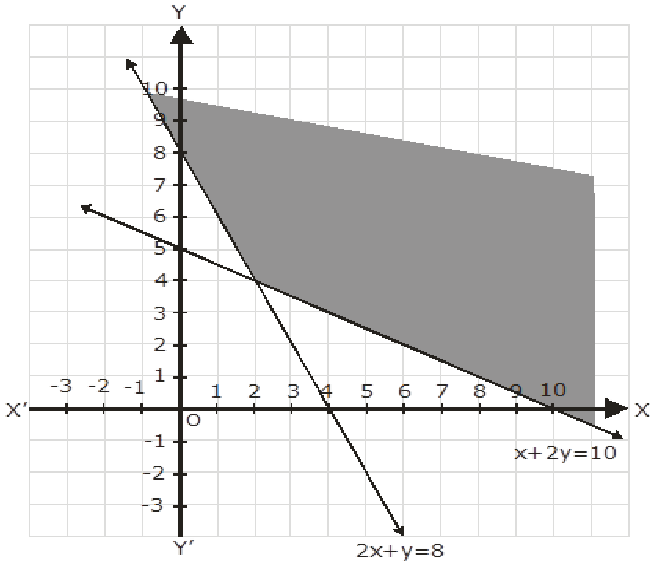
The graph is as follows:



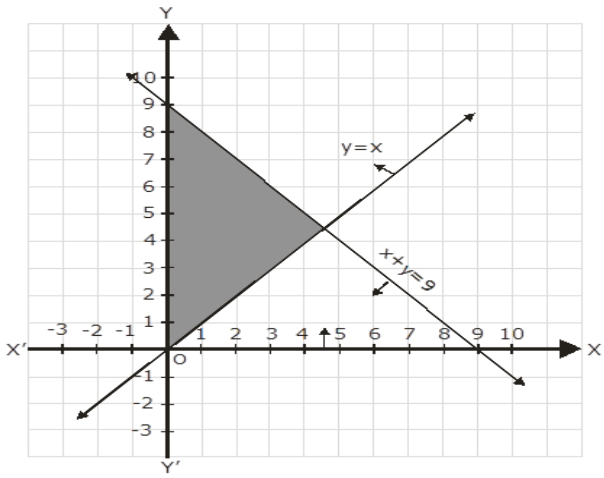
A.6.



A.7.


Y
10
9
8
7
6
5
4
3
2
1
-3 -2 -1
1 2 3 4 5 6 7 8 9 10
X'
X
O
-1
-2
-3
Y'
x+2y=10
2x+y=8

A.8.


Y
10
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5
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3
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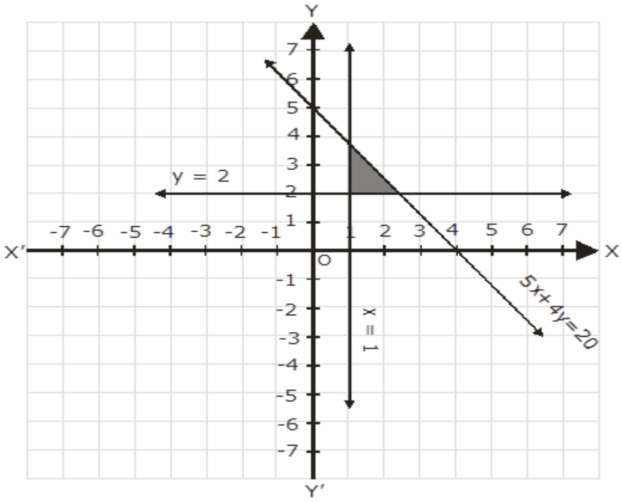
1
-3 -2 -1
1 2 3 4 5 6 7 8 9 10
X'
X
O
-1
-2
-3
Y'
y=x
x+y=9

A.9.



A.10. The given inequality is $3x + 4y \leq 60$,

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 15 and x = 20

The required points are (0, 15) and (20, 0)

Checking if the origin lies in the required solution area (0,0)

$0 \leq 60$, this is true.

Hence the origin would lie in the solution area of the line`s graph.

The required solution area would be on the left of the line`s graph.

$x + 3y \leq 30$,

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 10 and x = 30

The required points are (0, 10) and (30, 0)

Checking for the origin (0, 0)

$0 \leq 30$, this is true.

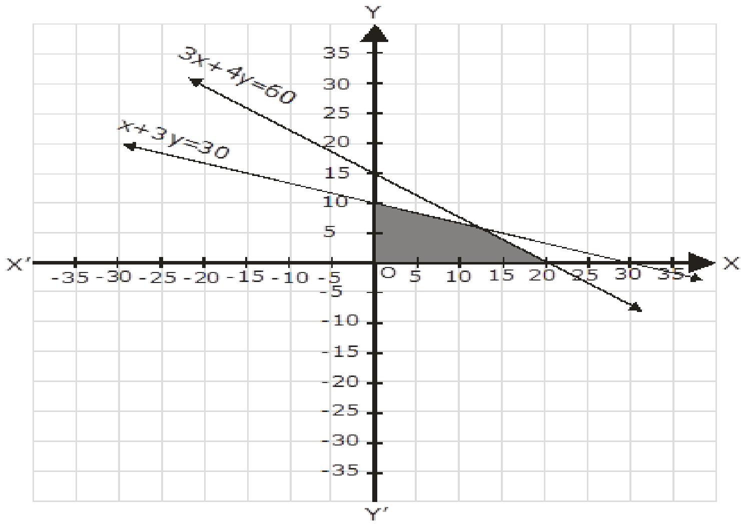
Hence the origin lies in the solution area which is given by the left side of the line`s graph.

$x \geq 0$,

$y \geq 0$,

The given inequalities imply the solution lies in the first quadrant only.

Hence the solution of the inequalities is given by the shaded region in the graph.



3.8 References/ Suggested Readings

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2. Dowling E.T.: Mathematics for Economics; Sihahum Series, McGraw Hill. London.
3. Kapoor, V. K.: Business Mathematics: Sultan, Chan & Sons, Delhi.
4. Loomba Paul: Linear Programming: Tata McGraw Hill, New Delhi.
5. Soni, R. S.: Business Mathematics: Pitamber Publishing House.

Lesson. 4 Linear Programming

Course Name: Business Mathematics

Course Code: BCOM 105

Semester-II

Author: Dr Vizender Singh

Structure:

4.0 Learning Objectives

The following are learning objective of Linear Programming Problems

- To determine the solution to a linear problem.
- To minimize or maximize the function such as such profit or cost functions.
- To analyze numerous economic, social, military and industrial problem.
- To improves quality of decision: A better quality can be obtained with the system by making use of linear programming.
- To unify results from disparate areas of mechanism design.
- To find the more flexible system, a wide range of problems can be solved easily.

4.1 Introduction

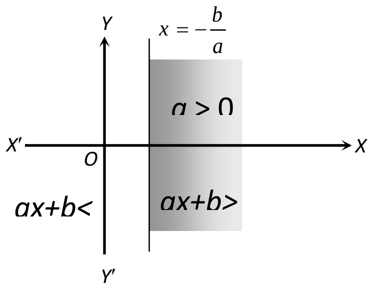
'Linear Programming' is a scientific tool to handle optimization problems. Here, we shall learn about some basic concepts of linear programming problems in two variables, their applications, advantages, limitations, formulation and graphical method of solution.

4.2 Linear Programming

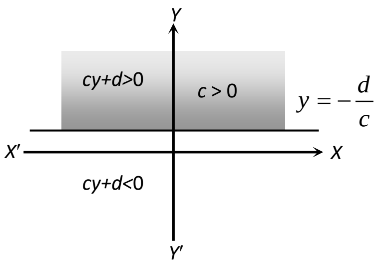
4.2.1 Linear inequations

(1) **Graph of Linear Inequations**

(i) **Linear inequation in one variable:** $ax+b>0$, $ax+b<0$, $cy+d>0$ etc., are called linear inequations in one variable. Graph of these inequations can be drawn as follows:

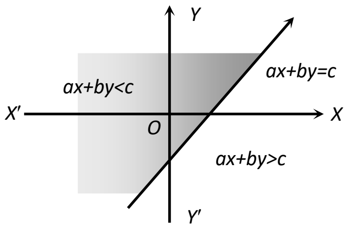


The graph of $ax+b>0$ and $ax+b<0$ are obtained by dividing *xy*-plane in two semi-planes by the line $x=-\frac{b}{a}$ (which is parallel to *y*-axis). Similarly for $cy+d>0$ and $cy+d<0$.



(ii) **Linear Inequation in two variables:** General form of these inequations are $ax+by>c, ax+by<c$. If any ordered pair (x_1, y_1) satisfies an inequation, then it is said to be a solution of the inequation.

The graph of these inequations is given below (for $c > 0$) :



Working Rule: To draw the graph of an inequation, following procedure is followed :

(i) Write the equation $ax + by = c$ in place of $ax + by < c$ and $ax + by > c$.

(ii) Make a table for the solutions of $ax + by = c$.

(iii) Now draw a line with the help of these points. This is the graph of the line $ax + by = c$.

(iv) If the inequation is > or <, then the points lying on this line is not considered and line is drawn dotted or discontinuous.

(v) If the ineuqation is ≥ or ≤, then the points lying on the line is considered and line is drawn bold or continuous.

(vi) This line divides the plane *XOY* in two region.

To Find the region that satisfies the inequation, we apply the following rules:

(a) Take an arbitrary point which will be in either region.

(b) If it satisfies the given inequation, then the required region will be the region in which the arbitrary point is located.

(c) If it does not satisfy the inequation, then the other region is the required region.

(d) Draw the lines in the required region or make it shaded.

(2) **Simultaneous linear inequations in two variables:** Since the solution set of a system of simultaneous linear inequations is the set of all points in two dimensional space which satisfy all the inequations simultaneously. Therefore to find the solution set we find the region of the plane common to all the portions comprising the solution set of given inequations. In case there is no region common to all the solutions of the given inequations, we say that the solution set is **void** or **empty**.

(3) **Feasible region:** The limited (bounded) region of the graph made by two inequations is called feasible region. All the points in feasible region constitute the solution of a system of inequations. The feasible solution of a L.P.P. belongs to only first quadrant. If feasible region is empty then there is no solution for the problem.

4.2.2 Terms of linear programming

The term programming means planning and refers to a process of determining a particular program.

(1) **Objective Function :** The linear function which is to be optimized (maximized or minimized) is called objective function of the *L.P.P*.

(2) **Constraints or Restrictions :** The conditions of the problem expressed as simultaneous equations or inequations are called constraints or restrictions.

(3) **Non-negative Constraints :** Variables applied in the objective function of a linear programming problem are always non-negative. The inequations which represent such constraints are called non-negative constraints.

(4) **Basic Variables :** The m variables associated with columns of the $m \times n$ non-singular matrix which may be different from zero, are called basic variables.

(5) **Basic Solution :** A solution in which the vectors associated to m variables are linear and the remaining $(n - m)$ variables are zero, is called a basic solution. A basic solution is called a degenerate basic solution, if at least one of the basic variables is zero and basic solution is called non-degenerate, if none of the basic variables is zero.

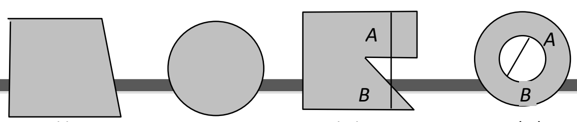
(6) **Feasible Solution:** The set of values of the variables which satisfies the set of constraints of linear programming problem (*L.P.P*) is called a feasible solution of the *L.P.P*.

(7) **Optimal Solution :** A feasible solution for which the objective function is minimum or maximum is called optimal solution.

(8) **Iso-Profit Line :** The line drawn in geometrical area of feasible region of *L.P.P*. for which the objective function (to be maximized) remains constant at all the points lying on the line, is called iso-profit line.

If the objective function is to be minimized then these lines are called **iso-cost** lines.

(9) **Convex set :** In linear programming problems feasible solution is generally a polygon in first quadrant. This polygon is convex. It means if two points of polygon are connected by a line, then the line must be inside the polygon. For example,



(i) (ii) (iii) (iv)

Fig. (i) and (ii) are convex set while (iii) and (iv) are not convex set.

4.2.3 Mathematical Formulation of a Linear Programming Problem

There are mainly four steps in the mathematical formulation of a linear programming problem, as mathematical model. We will discuss formulation of those problems which involve only two variables.

(1) Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.

(2) Identify the set of constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the given conditions.

(3) Identify the objective function and express it as a linear function of decision variables. It may take the form of maximizing profit or production or minimizing cost.

(4) Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

4.2.4 Graphical Solution of Two Variable Linear Programming Problem

There are two techniques of solving an *L.P.P.* by graphical method. These are :

(1) Corner point method (2) Iso-profit or Iso-cost method

(1) **Corner Point Method**

***Working Rule*:**

(i) Formulate mathematically the *L.P.P.*

(ii) Draw graph for every constraint.

(iii) Find the feasible solution region.

(iv) Find the coordinates of the vertices of feasible solution region.

(v) Calculate the value of objective function at these vertices.

(vi) Optimal value (minimum or maximum) is the required solution.

(vii) If there is no possibility to determine the point at which the suitable solution found, then the solution of problem is unbounded.

(viii) If feasible region is empty, then there is no solution for the problem.

(ix) Nearer to the origin, the objective function is minimum and that of further from the origin, the objective function is maximum.

(2) **Iso-Profit or Iso-Cost Method:** Various steps of the method are as follows:

(i) Find the feasible region of the *L.P.P.*

(ii) Assign a constant value Z_1 to Z and draw the corresponding line of the objective function.

(iii) Assign another value Z_2 to Z and draw the corresponding line of the objective function.

(iv) If $Z_1 < Z_2, (Z_1 > Z_2)$, then in case of maximization (minimization) move the line P_1Q_1 corresponding to Z_1 to the line P_2Q_2 corresponding to Z_2 parallel to itself as far as possible, until the farthest point within the feasible region is touched by this line. The coordinates of the point give maximum (minimum) value of the objective function.

(v) The problem with more equations/inequations can be handled easily by this method.

(vi) In case of unbounded region, it either finds an optimal solution or declares an unbounded solution. Unbounded solutions are not considered optimal solution. In real world problems, unlimited profit or loss is not possible.

4.2.5. To Find the Vertices of Simple Feasible Region without Drawing a Graph

(1) **Bounded Region:** The region surrounded by the inequations $ax + by \le m$ and $cx + dy \le n$ in first quadrant is called bounded region. It is of the form of triangle or quadrilateral. Change these inequations into equations, then by putting $x = 0$ and $y = 0,$ we get the solution. Also by solving the equations, we get the vertices of bounded region.

The maximum value of objective function lies at one vertex in limited region.

(2) **Unbounded Region:** The region surrounded by the inequations $ax + by \ge m$ and $cx + dy \ge n$ in first quadrant, is called unbounded region.

Change the inequation in equations and solve for $x = 0$ and $y = 0$. Thus we get the vertices of feasible region.

The minimum value of objective function lies at one vertex in unbounded region but there is no existence of maximum value.

4.2.6 Advantages and Limitations of L.P.P.

(1) **Advantages:** Linear programming is used to minimize the cost of production for maximum output. In short, with the help of linear programming models, a decision maker can most efficiently and effectively employ his production factor and limited resources to get maximum profit at minimum cost.

(2) **Limitations:** (i) The linear programming can be applied only when the objective function and all the constraints can be expressed in terms of linear equations/inequations.

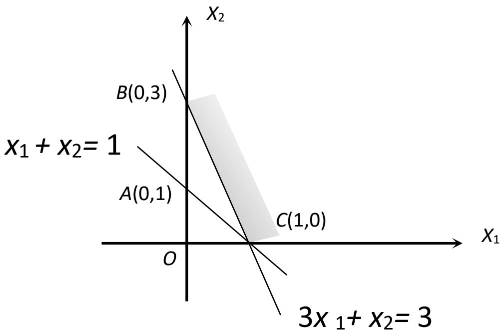
(ii) Linear programming techniques provide solutions only when all the elements related to a problem can be quantified.

(iii) The coefficients in the objective function and in the constraints must be known with certainty and should remain unchanged during the period of study.

(iv) Linear programming technique may give fractional valued answer which is not desirable in some problems.

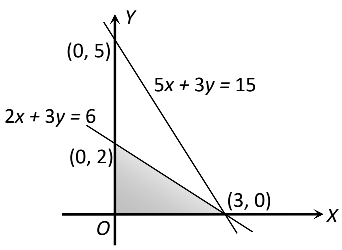
Example 4.2.7 Find the number of feasible regions for the constraint of a linear optimizing function $z = x_1 + x_2$, given by $x_1 + x_2 \leq 1, 3x_1 + x_2 \geq 3$ and $x_1, x_2 \geq 0$

Solution. Clearly from graph there is no feasible region.



Example 4.2.8 Find the points which are the vertices of the positive region bounded by the inequalities $2x + 3y \leq 6$, $5x + 3y \leq 15$ and $x, y \geq 0$.

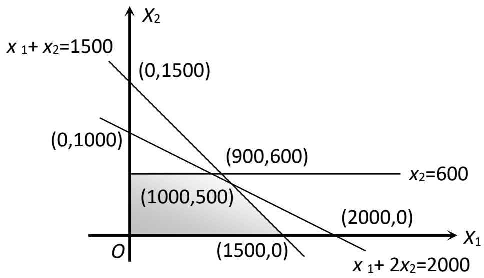
Solution. Drawing the graph, we have, the feasible region is the shaded are and



clearly (0, 2); (0, 0) and (3, 0) all are vertices of feasible region.

Example 4.2.9 For the constraints of a L.P.P. problem given by $x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 1500$, $x_2 \le 600$ and $x_1, x_2 \ge 0$, find the points which lies in the positive bounded region.

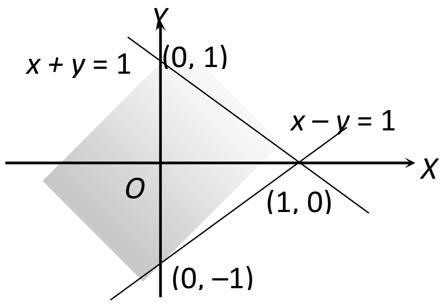
Solution. Drawing the graph, we have, the feasible region is the shaded are and



clearly the vertices, (0, 0), (0, 1000); (1000, 500) and (1500, 0) all lies in the positive bounded region.

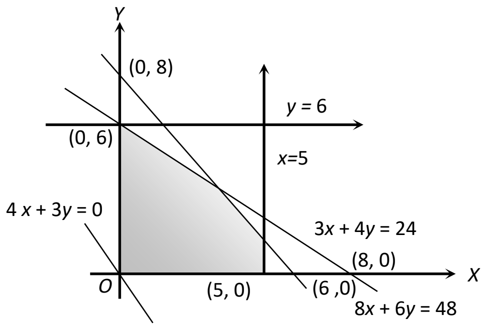
Example 4.2.10 Find the quadrant in which, the bounded region for inequations $x + y \le 1$ and $x - y \le 1$ is situated.

Solution. As shown in graph drawn for $x + y = 1$ and $x - y = 1$ the origin included in the area. Hence the bounded region situated in all four quadrant.



Example 4.2.11 Find the number of points at which the objective function $z = 4x + 3y$ can be maximized subjected to the constraints $3x + 4y \le 24$, $8x + 6y \le 48$, $x \le 5, y \le 6;\ x, y \ge 0$.

Solutiojn. Obviously, the optimal solution is found on the line which is parallel to 'isoprofit line'. Hence it has infinite number of solution.



Example 4.2.12 A wholesale merchant wants to start the business of cereal with *Rs*. 24000. Wheat is *Rs*. 400 per *quintal* and rice is Rs. 600 per *quintal*. He has capacity to store 200 *quintal* cereal. He earns the profit *Rs*. 25 per *quintal* on wheat and *Rs*. 40 per *quintal* on rice. If he stores *x quintal* rice and *y quintal* wheat, then for maximum profit find the objective function.

Solution. For maximum profit, $z = 40x + 25y$.

Example 4.2.13 Mohan wants to invest the total amount of *Rs*. 15,000 in saving certificates and national saving bonds. According to rules, he has to invest at least *Rs*. 2000 in saving certificates and *Rs*. 2500 in national saving bonds. The interest rate is 8% on saving certificate and 10% on national saving bonds per annum. He invest *Rs*. *x* in saving certificates and *Rs*. *y* in national saving bonds. Then write the objective function for this problem.

Solution. Objective function is given by profit function

$$Z = x \cdot \frac{8}{100} + y \times \frac{10}{100} = 0.08x + 0.10y .$$

Example 4.2.14 A firm produces two types of products *A* and *B*. The profit on both is *Rs*. 2 per item. Every product requires processing on machines M_1 and M_2. For *A*, machines M_1 and M_2 takes 1 *minute* and 2 *minute* respectively and for *B*, machines M_1 and M_2 takes the time 1 *minute* each. The machines M_1, M_2 are not available more than 8 *hours* and 10 *hours*, any of day, respectively. If the products made *x* of *A* and *y* of *B*, then find the linear constraints for the L.P.P. except $x \geq 0, y \geq 0$.

Solution. Obviously $x + y \leq (8 \times 60 = 480)$ and $2x + y \leq (10 \times 60 = 600)$.

Example 4.2.15 In a test of Mathematics, there is two types of questions to be answered–short answered and long answered. The relevant data is given below

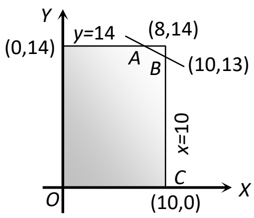
Type of questions	**Time taken to solve**	**Marks**	**Number of questions**
Short answered questions	5 Minutes	3	10
Long answered questions	10 Minutes	5	14

The total marks is 100. Students can solve all the questions. To secure maximum marks, a student solves *x* short answered and *y* long answered questions in three hours, then find the linear constraints except $x \geq 0, y \geq 0$.

Also, find vertices of a feasible region and maximum value of objective function.

Solution. Obviously linear constraints except are $x \leq 10, y \leq 14$ and $5x + 10y \leq 180$.

By drawing graph the required feasible region is given by $ABCD$, and vertices are (8, 14), (10, 13), (10, 0) and (0, 14)



And maximum value of objective function *Max* $z = 3(10) + 5(13) = 95$.

Example 4.2.16 A factory produces two products *A* and *B*. In the manufacturing of product A, the machine and the carpenter requires 3 *hour* each and in manufacturing of product *B*, the machine and carpenter requires 5 *hour* and 3 *hour* respectively. The machine and carpenter work at most 80 *hour* and 50 *hour* per week respectively. The profit on *A* and *B* is *Rs*. 6 and 8 respectively. If profit is maximum by manufacturing *x* and *y* units of *A* and *B* type product respectively, then for the function $6x + 8y$, write the constraints of problem.

Solution. The constraints of problem are $x, y \geq 0, 3x + 5y \leq 80, 3x + 3y \leq 50$.

Example 4.2.17 A shopkeeper wants to purchase two articles *A* and *B* of cost price *Rs*. 4 and 3 respectively. He thought that he may earn 30 *paise* by selling article *A* and 10 *paise* by selling article *B*.

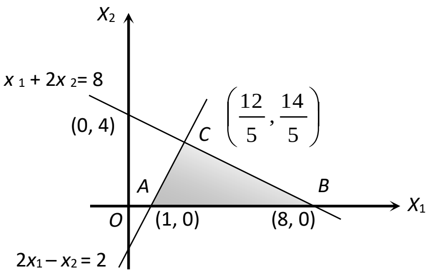
He has not to purchase total article worth more than *Rs*. 24. If he purchases the number of articles of *A* and *B*, *x* and *y* respectively, then find the linear constraints. Also write the profit line.

Solution. The constraints of problem are $x \geq 0, y \geq 0, 4x + 3y \leq 24$ and the profit line is

$$4x + 3y = 24$$

Example 4.2.18 For the L.P. problem $Max\, z = 3x_1 + 2x_2$ such that $2x_1 - x_2 \geq 2$, $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$, find z.

Solution. By Changing the inequalities into equations and drawing the graph of lines, we get the required feasible region.

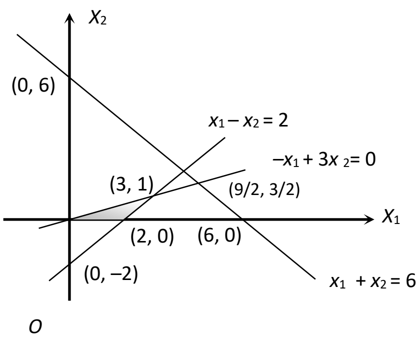


It is a bounded region, bounded by the vertices $A(1,0), B(8,0)$ and $C\left(\frac{12}{5}, \frac{14}{5}\right)$.

Now by evaluation of the objective function for the vertices of feasible region it is found to be maximum at (8,0). Hence the solution is $z = 3 \times 8 + 0 \times 2 = 24$.

Example 4.2.19 For the L.P. problem Min $z = -x_1 + 2x_2$ such that $-x_1 + 3x_2 \leq 0,\ x_1 + x_2 \leq 6,$ $x_1 - x_2 \leq 2$ and $x_1, x_2 \geq 0$, the find x_1.

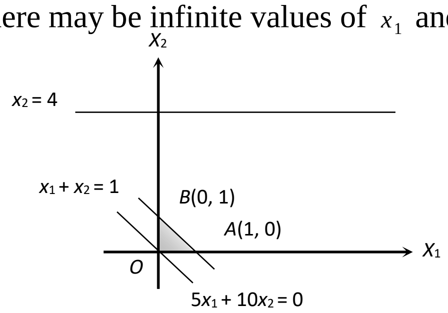
Solution. Here $(3,1), (2,0)$ are vertices of *Min z* for (2, 0)



Hence $x_1 = 2$.

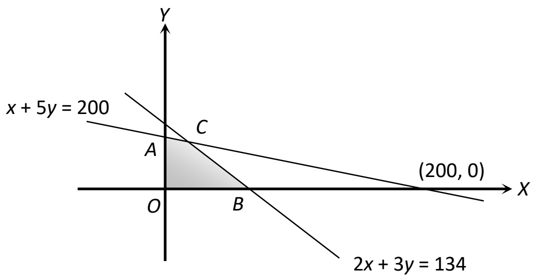
Example 4.2.20 For the L.P. problem $Min\, z = x_1 + x_2$ such that $5x_1 + 10x_2 \le 0,\ x_1 + x_2 \ge 1,\ x_2 \le 4$ and $x_1, x_2 \ge 0$, find the number of solutions for the problem.

Solution. As there may be infinite values of x_1 and x_2 on line $x_1 + x_2 = 1$.



Example 4.2.21 On maximizing $z = 4x + 9y$ subject to $x + 5y \le 200,\ \ 2x + 3y \le 134$ and $x, y \ge 0$, find z.

Solution. Here, $A = (0,40), B = (67,0)$ and $C = (10,38)$



Maximum for C *i.e.*, *z* = 40 + 342 = 382.

Example 4.2.22 For the L.P. problem $Min\, z = 2x + y$ subject to $5x + 10y \le 50,\ \ x + y \ge 1,\ y \le 4$ and $x, y \ge 0$, Find z.

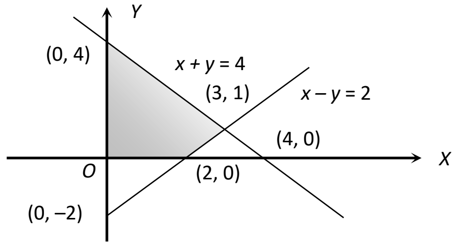
Solution. After drawing a graph, we get the vertices of feasible region are (1, 0), (10, 0), (2, 4), (0, 4) and (0, 1).

Thus minimum value of objective function is at (0, 1)

Hence $z = 0 \times 2 + 1 \times 1 = 1$.

Example 4.2.23 Find the solution of a problem to maximize the objective function $z = x + 2y$ under the constraints $x - y \le 2$, $x + y \le 4$ and $x, y \ge 0$.

Solution. Here $z = x + 2y$



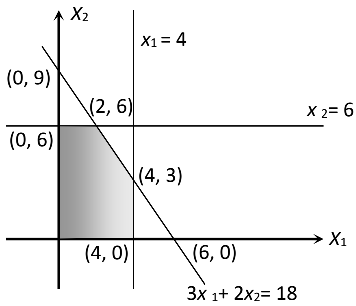
Max $z = 0 + 4(2) = 8$.

Example 4.2.24 By graphical method, find the the solution of linear programming problem

Maximize $z = 3x_1 + 5x_2$

Subject to $3x_1 + 2x_2 \le 18$, $x_1 \le 4$, $x_2 \le 6$, $x_1 \ge 0$, $x_2 \ge 0$.

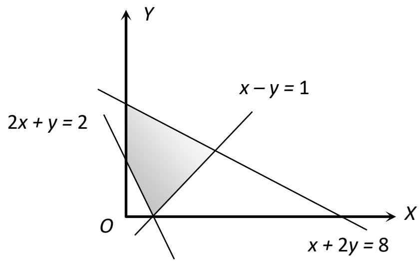
Solution. By drawing the graph, we have



Here feasible region has vertices (0, 0); (4, 0); (4, 3); (2, 6) and (0, 6).

$\therefore$ *Max z* at $(2,6) = 3(2) + 5(6) = 36$.

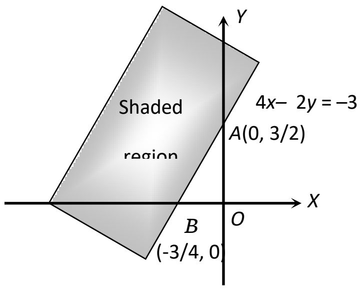
Example 4.2.25 For the following shaded area, find the linear constraints except $x \ge 0$ and $y \ge 0$.



Solution. To test the origin for $2x+y=2, x-y=1$ and $x+2y=8$ in reference to shaded area, $0+0<2$ is true for $2x+y=2$. So for the region does not include origin (0, 0), $2x+y\geq 2$. Again for $x-y=1, 0-0<1$, $\therefore\ x-y\leq 1$

Similarly for $x+2y=8, 0+0<8$; $\therefore\ x+2y\leq 8$.

Example 4.2.26 Write the inequation representing the shaded region



Solution. Origin is not present in given shaded area, so $4x-2y\leq -3$ satisfy this condition.

Example 4.2.27 A Firm makes pents and shirts. A shirt takes 2 hour on machine and 3 hour of man labour while a pent takes 3 hour on machine and 2 hour of man labour. In a week there are 70 hour machine and 75 hour of man labour available. If the firm determine to make *x* shirts and *y* pents per week, then for this L.P.P. write the linear constraints.

Solution. Here

Type of items	Working time on machine	Man labour

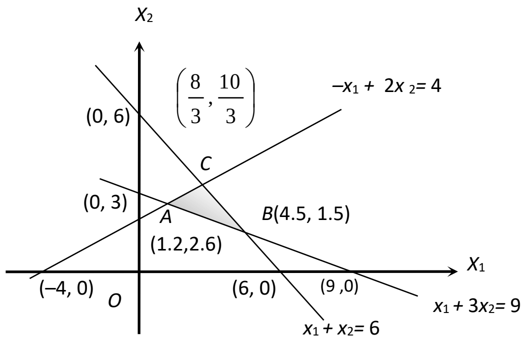
Shirt (x) Pent (y)	2 hours 3 hours	3 hours 2 hours
Availability	70 hours	75 hours

Linear constraints are $2x+3y \le 70, 3x+2y \le 75$.

Example 4.2.28 For the L.P. problem $Min\, z = 2x_1 + 3x_2$ such that $-x_1 + 2x_2 \le 4$,

$x_1 + x_2 \le 6,\ x_1 + 3x_2 \ge 9$ and $x_1, x_2 \ge 0$. Find variables and objective function z.

Solution. The graph of linear programming problem is as given below



Hence the required feasible region is given by the graph whose vertices are $A(1.2, 2.6), B(4.5,1.5)$ and $C\left(\frac{8}{3}, \frac{10}{3}\right)$.

Thus objective function is minimum at $A(1.2, 2.6)$

So $x_1 = 1.2, x_2 = 2.6$ and $z = 2\times 1.2 + 3\times 2.6 = 10.2$.

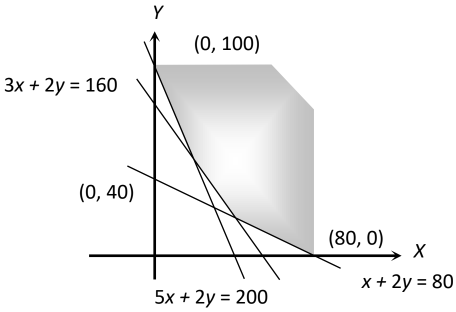
Example 4.2.29 A company manufactures two types of products *A* and *B*. The storage capacity of its godown is 100 units. Total investment amount is *Rs*. 30,000. The cost price of *A* and *B* are *Rs*. 400 and *Rs*. 900 respectively. If all the products have sold and per unit profit is *Rs*. 100 and *Rs*. 120 through *A* and *B* respectively. If *x* units of *A* and *y* units of *B* be produced, then find the two linear constraints and iso-profit line.

Solution. Here $x + y \le 100, 400x + 900y \le 30000$ or

$4x + 9y \le 300$ and $100x + 120y = c$.

Example 4.2.30 Find the the maximum value of $z = 4x + 3y$ subject to the constraints $3x + 2y \ge 160, 5x + 2y \ge 200$, $x + 2y \ge 80$; $x, y \ge 0$

Solutions. Obviously, it is unbounded. Therefore its maximum value does not exist.



Example 4.2.31 Minimize $z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$

Subject to : $\sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1,......., m$

$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1,......, n$

is a (L.P.P.) with number of constraints as $m + n$.

Solution. Here

$i = 1, x_{11} + x_{12} + x_{13} + ... + x_{1n}$

$i = 2, x_{21} + x_{22} + x_{23} + ... + x_{2n}$

$i = 3, x_{31} + x_{32} + x_{33} + ... + x_{3n}$

....

$i = m, x_{m1} + x_{m2} + x_{m3} + ... x_{mn} \rightarrow$constraints

Condition (ii),

$j = 1, x_{11} + x_{21} + x_{31} + ... + x_{m1}$

$j = 2, x_{12} + x_{22} + x_{32} + ... + x_{m1}$

....

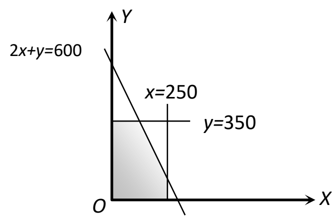
$j = n, x_{1n} + x_{2n} + x_{3n} + ... + x_{mn} \rightarrow n$ constraints

$\therefore$ Total constraints = $m + n$.

4.3 Check Your Progress

Answer the following objective type questions.

Q.1. For the following feasible region, the linear constraints except $x \ge 0$ and $y \ge 0$, are



(a) $x \geq 250, y \leq 350, 2x + y = 600$

(b) $x \leq 250, y \leq 350, 2x + y = 600$

(c) $x \leq 250, y \leq 350, 2x + y \geq 600$

(d) $x \leq 250, y \leq 350, 2x + y \leq 600$

Q.2. Let X_1 and X_2 are optimal solutions of a L.P.P., then

(a) $X = \lambda X_1 + (1-\lambda)X_2, \lambda \in R$ is also an optimal solution

(b) $X = \lambda X_1 + (1-\lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution

(c) $X = \lambda X_1 + (1+\lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution

(d) $X = \lambda X_1 + (1+\lambda)X_2, \lambda \in R$ gives an optimal solution

Q.3. The points which provides the solution to the linear programming problem: $Max(2x + 3y)$ subject to constraints: $x \geq 0, y \geq 0,\ 2x + 2y \leq 9,\ 2x + y \leq 7,\ x + 2y \leq 8$, is

(a) (3, 2.5) (b) (2, 3.5)

(c) (2, 2.5) (d) (1, 3.5)

Q.4. Two tailors *A* and *B* earns *Rs.* 15 and *Rs.* 20 per day respectively. A can make 6 shirts and 4 pents in a day while *B* can make 10 shirts and 3 pents. To spend minimum on 60 shirts and 40 pents, *A* and *B* working *x* and *y* days respectively. Then linear constraints except $x \geq 0, y \geq 0$, are and objective function are respectively

(a) $15x + 20y \geq 0, 60x + 40y \geq 0$, $z = 4x + 3y$

(b) $15x + 20y \geq 0, 6x + 10y = 10$, $z = 60x + 60y$

(c) $6x + 10y \geq 60, 4x + 3y \geq 40$, $z = 60x + 40y$

(d) $6x + 10y \leq 60, 4x + 3y \leq 40$, $z = 15x + 20y$

Q.5. A company manufactures two types of telephone sets *A* and *B*. The *A* type telephone set requires 2 hour and *B* type telephone requires 4 hour to make. The company has 800 work hour per day. 300 telephone can pack in a day. The selling prices of *A* and *B* type telephones are *Rs*. 300 and 400 respectively. For maximum profits company produces *x* telephones of *A* type and *y* telephones of *B* types. Then except $x \geq 0$ and $y \geq 0$, linear constraints and the probable region of the L.P.P is of the type

(a) $x + 2y \leq 400; x + y \leq 300$; $Max\, z = 300x + 400y$, bounded

(b) $2x + y \leq 400; x + y \geq 300$; $Max\, z = 400x + 300y$, unbounded

(c) $2x + y \geq 400; x + y \geq 300$; $Max\, z = 300x + 400y$, parallelogram

(d) $x + 2y \leq 400; x + y \geq 300$; $Max\, z = 300x + 400y$, square

Q.6. We have to purchase two articles *A* and *B* of cost *Rs*. 45 and *Rs*. 25 respectively. I can purchase total article maximum of *Rs*. 1000. After selling the articles *A* and *B*, the profit per unit is *Rs*. 5 and 3 respectively. If I purchase the *x* and *y* numbers of articles *A* and *B* respectively, then the mathematical formulation of problem is

(a) $x \geq 0, y \geq 0,\ 45x + 25y \geq 1000,\ 5x + 3y = c$

(b) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000,\ 5x + 3y = c$

(c) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000,\ 3x + 5y = c$

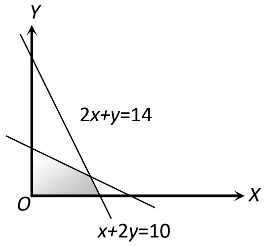
(d) None of these

Q.7. For the L.P. problem $Max\, z = 3x + 2y$ subject to $x + y \geq 1,\ y - 5x \leq 0,\ x - y \geq -1,\ x + y \leq 6,\ x \leq 3$ and $x, y \geq 0$

(a) $x = 3$ (b) $y = 3$

(c) $z = 15$ (d) All the above

Q.8. The maximum value of objective function $c = 2x + 3y$ in the given feasible region, is



(a) 29

(b) 18

(c) 14

(d) 15

Q.9. The maximum value of $4x+5y$ subject to the constraints $x+y\le 20,\ x+2y\le 35,\ x-3y\le 12$ is

(a) 84 (b) 95

(c) 100 (d) 96

Q.10. For the following linear programming problem: minimize $z=4x+6y$ subject to the constraints $2x+3y\ge 6$, $x+y\le 8,\ \ y\ge 1, x\ge 0$, the solution is

(a) (0, 2) and (1, 1) (b) (0, 2) and $(3/2, 1)$

(c) (0, 2) and (1, 6) (d) (0, 2) and (1, 5)

4.4 Summary

- **In some of the linear programming problems, constraints are inconsistent *i.e.* there does not exist any point which satisfies all the constraints. Such type of linear programming problems are said to have *infeasible solution*.**
- **If the constraints in a linear programming problem are changed, the problem is to be re-evaluated.**
- **The optimal value of the objective function is attained at the point, given by corner points of the feasible region.**

- **If a *L.P.P.* admits two optimal solutions, it has an infinite number of optimal solutions.**
- **If there is no possibility to determine the point at which the suitable solution can be found, then the solution of problem is unbounded.**
- **The maximum value of objective function lies at one vertex in limited region.**

4.5 Keywords

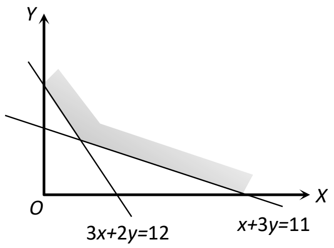
Linear equations and in equations, coordinate geometry, maxima and minima, profit and cost function.

4.6 Self-Assessment Test

Q.1. Find the vertex of common graph of inequalities $2x + y \geq 2$ and $x - y \leq 3$.

Q.2. Write the necessary condition for third quadrant region in *xy*-plane.

Q.3. For the following feasible region, find the linear constraints.



Q.4. Find the number of points at which the objective function $z = 4x + 3y$ can be maximized subjected to the constraints $3x + 4y \leq 24$, $8x + 6y \leq 48$, $x \leq 5, y \leq 6;$ $x, y \geq 0$.

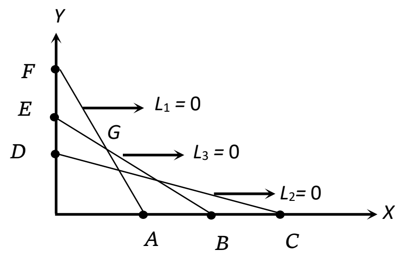
Q.5. Write the type of feasible region on which constraints

$$-x_1 + x_2 \leq 1$$
$$-x_1 + 3x_2 \leq 9$$

$x_1, x_2 \geq 0$ are defined.

Q.6. The quadrant in which the graph of inequations $x \leq y$ and $y \leq x + 3$ is located.

Q.7. The feasible region for the following constraints $L_1 \leq 0, L_2 \geq 0, L_3 = 0, x \geq 0, y \geq 0$ in the diagram shown is



Q.8. The sum of two positive integers is at most 5. The difference between two times of second number and first number is at most 4. If the first number is x and second number$y$, then for maximizing the product of these two numbers, write the mathematical formulation og L.P.P.

Q.9. Find the number of solutions for the L.P. problem $Min\, z = x_1 + x_2$ such that $5x_1 + 10x_2 \leq 0,\ x_1 + x_2 \geq 1,\ x_2 \leq 4$ and . $x_1, x_2 \geq 0$

Q.10. Find the maximum value of $P = 6x + 8y$ subject to constraints $2x + y \leq 30, x + 2y \leq 24$ and $x \geq 0, y \geq 0$.

Q.11. Find the maximum value of $P = x + 3y$ such that $2x + y \leq 20$, $x + 2y \leq 20$, $x \geq 0, y \geq 0$.

Q.12. Find the coordinate ofpoint at which the maximum value of $(x + y)$ subject to the constraints $2x + 5y \leq 100$, $\frac{x}{25} + \frac{y}{49} \leq 1$, $x, y \geq 0$ is obtained.

Q.13. Write the minimum value of $z = 2x_1 + 3x_2$ subject to the constraints

$2x_1 + 7x_2 \geq 22$, $x_1 + x_2 \geq 6$, $5x_1 + x_2 \geq 10$ and $x_1, x_2 \geq 0$.

Q.14. Find the co-ordinates of the point for minimum value of $z = 7x - 8y$ subject to the conditions $x + y - 20 \leq 0$, $y \geq 5,\ \ x \geq 0$, $y \geq 0$.

Q.15. The maximum value of $\mu = 3x + 4y$, subject to the conditions

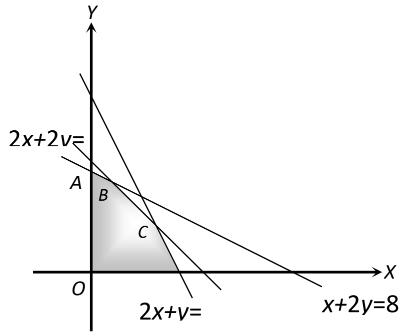
$x + y \leq 40, x + 2y \leq 60, x, y \geq 0$.

Answers

1	$\left(\frac{5}{3}, -\frac{4}{3}\right)$	2	$x < 0, y < 0$	3	$x \geq 0, y \geq 0,$ $x + 3y \geq 11$ $3x + 2y \geq 12,$	4	At an infinite number of points.	5	Both bounded and unbounded feasible space
6	I, II and III quadrants	7	Line segment *EG*	8	$x + y \leq 5$ $2y - x \leq 4,$ $x \geq 0, y \geq 0$	9	There are infinite solutions	10	120
11	30	12	$\left(\frac{50}{3}, \frac{40}{3}\right)$	13	14	14	(0, 20)	15	140

4.7 Answers to check your progress

1. (d) It is obvious.
2. (b) It is a fundamental concept.
3. (d) Given, $P = 2x + 3y$ Graph has been shown by given constraints and maximum value of *P* can be on *A* or *B* or *C* or *D*.



$P_A = P_{(0,4)} = 2(0) + 3(4) = 12$

$P_B = P_{(1,3.5)} = 2 \times 1 + 3 \times 3.5 = 12.5$

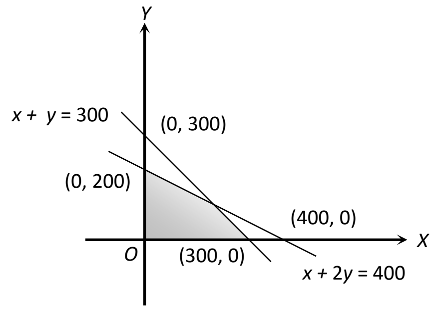
$P_C = P_{(2.5,2)} = 2 \times 2.5 + 3 \times 2 = 11$

$P_D = P_{(3.5,0)} = 2 \times 3.5 + 3 \times 0 = 7$

Obviously, at (1,3.5), $P_{\max} = 12.5$

4. (c) $6x + 10y \geq 60, 4x + 3y \geq 40$ (As minimum expenditure is concerned).

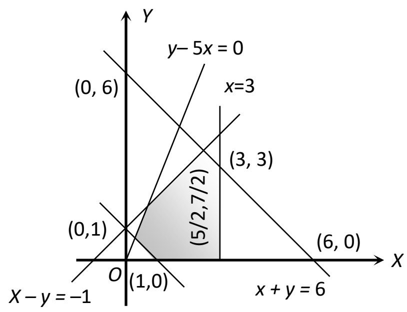
5. (a) The linear constraints are $x + 2y \leq 400, x + y \leq 300$ and $x, y \geq 0$. Also $Max\, z = 300x + 400y$.



Hence the region is bounded.

6. (b) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000, 5x + 3y = c$.

7. (d) The shaded region represents the bounded region.



Now, we get the maximum value of z at vertex (3, 3). So $Max\, z = 3(3) + 2(3) = 15$.

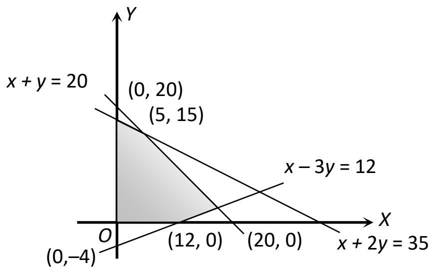
8. (b) The two vertices of given feasible region are (0, 5) and (7, 0) and third vertex can be found by solving the equations $x + 2y = 10$ and $2x + y = 14$, we get (6, 2) Now at (0, 5) $c = 2 \times 0 + 5 \times 3 = 15$, at (7, 0)

$c = 2 \times 7 + 0 \times 3 = 14$ and at (6, 2),

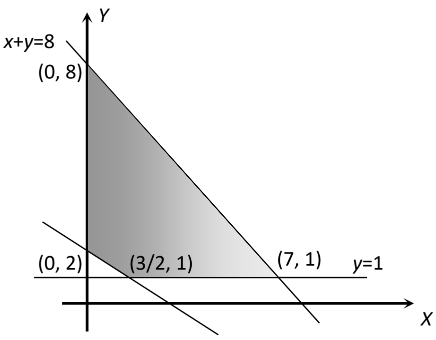
$c = 2 \times 6 + 3 \times 2 = 18$

Hence maximum value of objective function $c = 2x + 3y$ is 18 at point (6, 2).

9. (b) Obviously, $Max(4x + 5y) = 95$. It is at (5, 15).



10. (b)



Obviously, at $\left(\frac{3}{2},1\right)$ and (0, 2), $Min\, z = 4x + 6y = 12$.

4.8 References/ Suggested Readings

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Business Mathematics-II	
Lesson No: 5	**Author: Dr. Vizender Singh**
Data Representation and Interpretation I: Introduction, Classification and Tabulation of Data	**Vetter: Prof. Kuldeep Bansal**

Lesson Structure

5.0 LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Classification of data for further analysis
- The techniques of classification of data with their benefits
- The presentation of data through tables with their uses and drawbacks

5.1 INTRODUCTION

We can feel a lot of data around us every day and even every second of our life. These data does not make any sense until and unless you represent these data in a systematic manner. Whenever we try to collect data then we have to be prepared for the methods to classify these data. Data may be collected from the actual field of inquiry. For this purpose one may issue suitable questionnaires to get necessary information or he may take actual interviews; personal interviews are more effective than questionnaires. Further, data may be collected with the help of data available in publications of Government bodies or other public or private organizations. Such data are so large that one’s mind can hardly comprehend its significance in the form that it is shown. Therefore it becomes, very necessary to tabulate and summarize the data to an easily manageable form. Because statistics is not about an individual rather than it depends on aggregates of facts. In this chapter, we will discuss about classification and tabular presentation of data.

5.2 CLASSIFICATION OF DATA

Classification of data means the process of organizing data into relevant categories which may be used in an efficient manner. This classification of data is helpful to make data easy to locate and retrieve which leads to particular importance to risk managers and data security agents. The process of arranging data into homogenous groups or classes according to some common characteristics present in the data is called classification.

For example: During the process of sorting letters in a post office, the letters are classified according to the cities and further arranged according to streets.

Definition:

"Classified and arranged facts speak of themselves, and narrated they are as dead as mutton". --------------J.R. Hicks.

According to Horace Secrist,

"Classification is the process of arranging data into sequence and groups according to their common characteristics or separating them into different but related parts.

All similar items of data are put in one class and all dissimilar items of data are put in different classes. Statistical data is classified according to its characteristics. For example, if we have collected data regarding the number of students admitted to a university in a year, the students can be classified on the basis of sex. In this case, all male students will be put in one class and all female students will be put in another class. The students can also be classified on the basis of age, marks, marital status, height, etc. The set of characteristics we choose for the classification of the data depends upon the objective of the study. For example, if we want to study the religions mix of the students, we classify the students on the basis of religion.

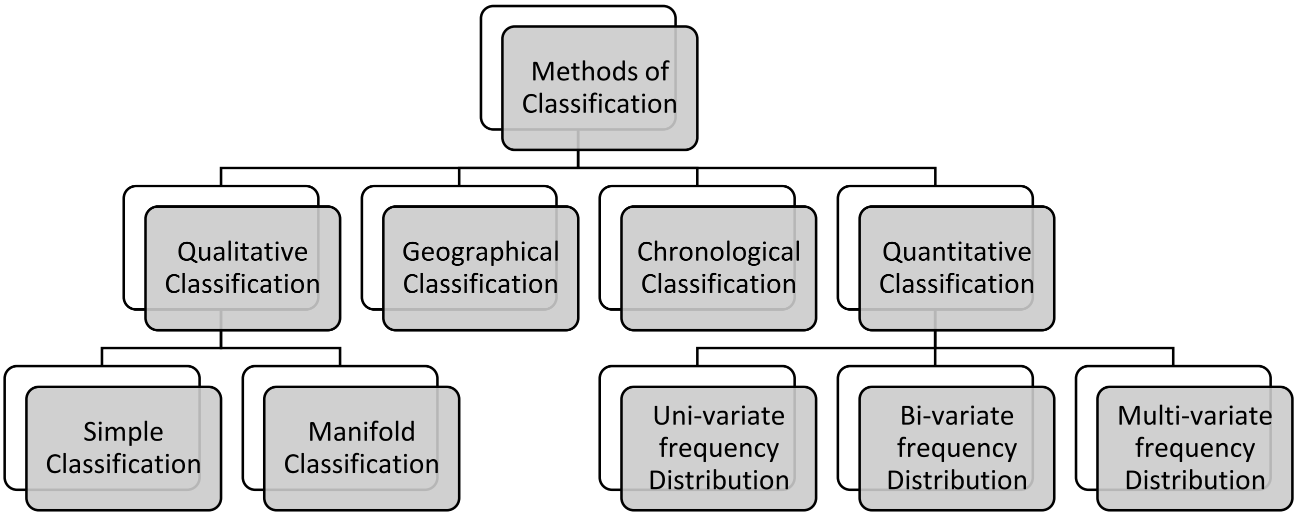
5.2.1 OBJECTIVES OF CLASSIFICATION OF DATA

The primary objectives of data classification are:

1. Data can be arranged in such a way that similarities and differences can be easily understood.
2. One can quickly sort data according to use.
3. Important features of data can be obtained at a glance.
4. It becomes easy to allow statistical tool when data are classified.
5. Classification is an important element for tabulation because tabulation is not possible without classification of data.
6. Classification of data makes data attractive and effective.

5.2.2 METHODS OF CLASSIFICATION OF DATA

There are four basis of classification of data accordance with their characteristics which are described as under:

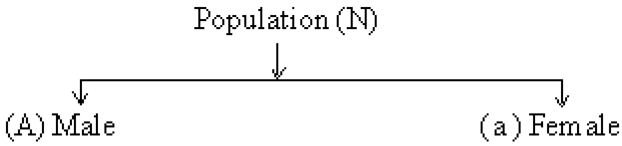


1. Qualitative Classification

Data are classified on the basis of some qualities or attributes such as honesty, beauty, intelligence, literacy, marital status, caste, sex, etc. An attribute is a qualitative characteristic which cannot be expressed numerically. Only the presence or absence of an attribute can be known. When classification is to be done on the basis of attributes, groups are differentiated either by the presence or absence of the attribute or by its differing qualities. For example, if we select colour of hair as the basis of classification, there will be a group of brown haired people and another group of black haired people. There are two types of classification based on features or attributes.

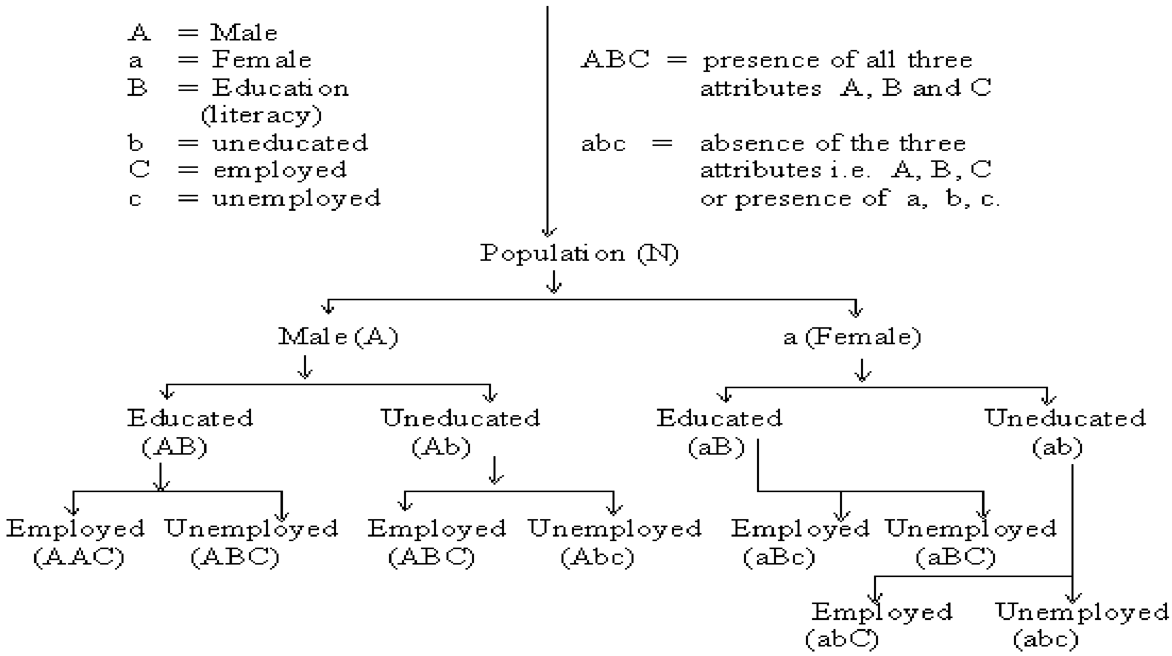
Simple Classification

In simple classification the data is classified on the basis of only one attribute. For example: classification of population according to sex i.e. male or female is known as simple classification.



Manifold Classification

In this classification the data is classified on the basis of more than one attribute. For example, the data relating to the number of students in a university can be classified on the basis of their sex i.e. male or female and then further classified on the basis of marital status i.e. married and unmarried.



2. **Geographical Classification**

When data are classified with reference to geographical locations such as countries, states, cities, districts, etc. it is known as Geographical Classification. It is also known as ‘Spatial Classification’.

Number of Firms Producing Car in 2019

State	**No. of Firms**

Punjab	30
Haryana	20
Rajasthan	25
U.P.	35

3. **Chronological Classification**

When data are grouped according to time, such a classification is known as a Chronological Classification. In a chronological classification, data are classified either in ascending or in descending order with reference to time such as years, quarters, months, weeks, etc. It is also called 'Temporal Classification'.

Population of India (1951 to 1991)

Year	**Population (in Crores)**
1951	36.1
1961	43.9
1971	54.8
1981	68.4
1991	84.4

4. **Quantitative Classification**

This type of classification is made on the basis some measurable or numerical characteristics like height, weight, age, income, marks of students, etc. this classification is done on the basis of some variables. Variables refer to quantifiable characteristics of data and can be expressed numerically. In this form of classification, the data is shown in the form of a frequency distribution. A frequency distribution is a tabular Presentation that generally organises data into classes, and shows the number of observations falling into each of these classes. Based on the number of variables used, there are three categories of frequency distribution.

Uni-variate Frequency Distribution

Any frequency distribution with one variable is known as uni-variate frequency distribution. For example, the students in a class may be classified on the basis of marks obained by them.

Bi-variate Frequency Distribution

Any frequency distribution with two variable is called bi-variate frequency distribution. If a frequency distribution shows two variables it is known as bi-variate frequency distribution.

Multi-variate Frequency Distribution

Any frequency distribution with more than two variables is called multivariate frequency distribution. For example, the students in a class may be classified on the basis of marks, age and sex.

DIFFERENT WAYS TO CLASSIFY NUMERICAL DATA

Numerical data are presented with the help of various types of series which are discussed as under:

I. Individual Series

Individual Series is a statistical series in which the all the observations are listed out and these observations have a single frequency. We can describe individual series as a series where each value of variable occurs only once. Further, such series are displayed without the frequency column. The following are the examples of individual series.

Example

Marks	No.1 Students
50	1
60	1

70	1
80	1
90	1

An individual series may be arranged either in ascending, or in descending, or in any other orders as it would suit the desired analysis. Like in the examples above, the series has been arranged in ascending order.

II. Frequency Distribution

Frequency distribution refers to the statistical table where values of variables are arranged in some magnitude and their corresponding frequencies are also shown side by side. There are two types of series under frequency distribution i.e. described as under:

(i) Discrete Series

It is a statistical table where individual values of variables are shown with their corresponding frequencies side by side. It is a simple and easy to understand series because frequencies are made with the counting of same variable in the series. These frequencies are shown with the help of tally bars which are also known as "Four and Cross Method". For example:

Consider the raw data which gives the size of shoes of 30 persons.

2,	5,	6,	4,	5,	7,	4,	4,	6,	2
3,	5,	5,	4,	5,	6,	5,	4,	3,	2
4,	4,	5,	4,	5,	5,	3,	2,	4,	4

The least value is 2 and the highest is 7. All sizes are integers between 2 and 7 (both inclusive). We can prepare a frequency distribution table as follows :

Table

Sizes of shoes	Tally Marks	Frequency
2	||||	4
3	|||	3
4	卌 卌	10
5	卌 卌	9
6	|||	3
7	|	1
Total		30

(ii) Continuous Series

It is also a statistical table where variables are shown in a group with their corresponding frequencies side by side. Further, such series can be stated either in the form exclusive, or in the form of inclusive class intervals along with their respective class frequencies. Furthermore, such series can also be presented either in non-cumulative, or in cumulative from (Less than, or more than) along with their respective frequencies. For example,

Daily Wages	**No. of Workers**
40-50	7
50-60	12
60-70	8
70-80	6
80-90	2
Total	35

Further, continuous series can be divided into major 5 parts i.e.

Exclusive Series

In exclusive series, the upper limit of a class becomes the lower limit of the next class. Here, we do not put any item that is equal to the upper limit of a class in the same class; we put it in the next class, i.e. the upper limits of classes are excluded from them.
For example, a person of age 20 years will not be included in the class-interval (10 - 20) but taken in the next class (20 - 30), since in the class interval (10 - 20) only units ranging from 10 - 19 are included. The exclusive-types of class-intervals can also be expressed as :

0 and below 10 or 0 - 9.9
10 and below 20 or 10 - 19.9
20 and below 30 or 20 - 29.9 and so on.

Inclusive Series

In inclusive series, the upper limit of any class interval is kept in the same class-interval. Here, the upper limit of a previous class is less by 1 from the lower limit of the next class interval. In short, this method allows a class-interval to include both its lower and upper limits within it. For example:

Table -

Inclusive method		Inclusive method	
Class	Frequency	Class	Frequency
0 - 4	5	0 - 4.9	5
5 - 9	7	5 - 9.9	7
10 - 14	9	10 - 14.9	9
15 - 19	12	15 - 19.9	12
20 - 24	11	20 - 24.9	11
25 - 29	14	25 - 29.9	14

Open ended Series

In any question when the lower limit of the first class-interval or the upper limit of the last class-interval, are not given then subtract the class length of the next immediate class-interval from the upper limit. This will give us the lower limit of the first class-interval. Similarly add the same class length to the lower limit of the last class-interval. But always notice that the lower limit of the first class (i.e. the lowest class) must not be negative or less than 0.
For example:

Table-

With open ends	Completed classes	With open ends	Completed classes
Below 10	0 - 10	Below 10	0 - 10
10 - 20	10 - 20	10 - 25	10 - 25
20 - 30	20 - 30	25 - 40	25 - 40
30 - 40	30 - 40	40 - 70	40 - 70
40 - 50	40 - 50	above 70	70 - 100
above 50	50 - 60		

Mid-value Series

We have only mid-values of the class intervals with their corresponding frequencies. This can be shown through an example,

Mid-value	5	15	25	35	45
Frequency	6	5	11	9	8

Mid-value Series can be converted into simple frequency series using some steps i.e. first of all, we have to determine the difference between mid-values then this difference is reduced to half which when deducted from the mid-value gives lower limit of the class interval and when added to the mid-value gives the corresponding upper limit. Thus,

$L_1 = m - i/2$

$L_2 = m + i/2$

Here, m = mid-value; i = difference between mid-values;

l_1 = lower limit and l_2 = upper limit

Frequency distribution with mid-values

Mid-values	Frequency	Class	Technique
5	6	0-10	L_1= 5-10/2=0, L_2 = 5 + 10/2 = 10
15	5	10-20	L_1= 15-10/2=10, L_2 = 15 + 10/2 = 20

25	11	20-30	L_1= 25-10/2=20, L_2 = 25 + 10/2 = 30
35	9	30-40	L_1= 35-10/2=30, L_2 = 35 + 10/2 = 40
45	8	40-50	L_1= 45-10/2=40, L_2 = 45 + 10/2 = 50

Cumulative Frequency Series

Many a times the frequencies of different classes are not given. Only their cumulative frequencies are given. The total frequency of all values less than or equal to the upper class boundary of a given class-interval is called the cumulative frequency up to and including that class interval. In this situation both the limits of a class-interval are not written; either lower or upper limit is written. These cumulative frequencies are called less than or more than cumulative frequencies. For example,

Class-interval	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency	4	9	5	12	15

Table -

Less than cumulative frequency		More than cumulative frequency	
(Upper limits)	(cum. freq.)	(Lower limits)	(cum. freq.)
Less than 10	4	More than 0	45 = 41 + 4
Less than 20	4 + 9 = 13	More than 10	41 = 32 + 9
Less than 30	13 + 5 = 18	More than 20	32 = 27 + 5
Less than 40	18 + 12 = 30	More than 30	27 = 15 + 12
Less than 50	30 + 15 = 45	More than 40	15

Some examples,

Example 1: The following gives the weights (to the nearest pounds) of 40 students of the 7th grade. Construct a frequency distribution table giving frequencies and cumulative frequencies of two types. Tabulate the data into equal classes, the 1st class being 118 - 122. Also indicate the class boundaries and the class-marks.

144, 125, 149, 157, 132, 150, 164, 138,

146, 158, 140, 147, 168, 126, 138, 176,

136, 148, 142, 144, 163, 119, 154, 165,

146, 173, 161, 145, 135, 142, 142, 147,

135, 153, 140, 135, 150, 156, 128, 145

Solution : The successive classes are 118 - 122, 123 - 127, ..., 173 - 177. The lower and upper class boundaries for the 1st class will be 117.5 and 122.5 and the class-mark will be

[redacted]

The class-boundaries and the class-marks will go on increasing by 5.

Table -

Class Boundaries	Class-marks	Tally-marks	Frequency	Cumulative frequency (less than)	Cumulative frequency (more than)	Relative frequency
117.5 - 122.5	120	|	1	1	39 + 1 = 40	2.5%
122.5 - 127.5	125	||	2	1 + 2 = 3	37 + 2 = 39	5%
127.5 - 132.5	130	||	2	3 + 2 = 5	35 + 2 = 37	5%
132.5 - 137.5	135	||||	4	5 + 4 = 9	31 + 4 = 35	10%
137.5 - 142.5	140	卌 |	6	9 + 6 = 15	25 + 6 = 31	15%
142.5 - 147.5	145	卌 |||	8	15 + 8 = 23	17 + 8 = 25	20%
147.5 - 152.5	150	卌	5	23 + 5 = 28	12 + 5 = 17	12.5%
152.5 - 157.5	155	||||	4	28 + 4 = 32	8 + 4 = 12	10%
157.5 - 162.5	160	||	2	32 + 4 = 34	6 + 2 = 8	5%
162.5 - 167.5	165	|||	3	34 + 3 = 37	3 + 3 = 6	7.5%
167.5 - 172.5	170	|	1	37 + 1 = 38	2 + 1 = 3	2.5%
172.5 - 177.5	175	||	2	38 + 2 = 40	2	5%
Total			$\Sigma f_i = 40$			100%

Example 2: Make a table showing the frequencies, cumulative frequencies (both less than and more than type) for the words having different numbers of letters in the following passage (ignore the punctuation marks).

"In Asia the complications are likely to be greater still. There will be four major powers in the Asian balance: Russia, China, Japan and the United States. To some extent, this increases the chance that the forces will emerge sufficient to contain China and it may lead to a real re-approachment between Russia and the West. But who knows, with so many factors, what the pattern will be like, ten years from now?"

Solution: We find that the smallest word ' a ' has one letter; and the longest word "re-approachment" has 14 letters. We prepare a tally sheet by writing the number of letters in the first column from 1 to 14. The table is as follows:

Table - 9

Number of letters	Tally-marks	Total
1	|	1
2	~~||||~~ ~~||||~~ |	11
3	~~||||~~ ~~||||~~ ~~||||~~ |	16
4	~~||||~~ ~~||||~~ ~~||||~~ |	16
5	~~||||~~ ||||	9
6	~~||||~~ ~~||||~~	10
7	~~||||~~ |	6
8	_	0
9	|	1
10	|	1
11	_	0
12	_	0
13	|	1
14	|	1
Total		73

Table-

Number of letters (xi)	Frequency	Cumulative frequency (< or =, type)	Cumulative frequency (> or =, type)
1	1	1	73
2	11	12	72
3	16	28	71
4	16	44	71
5	9	53	71
6	10	63	70
7	6	69	69
8	0	69	69
9	1	70	63
10	1	71	53
11	0	71	44
12	0	71	28
13	1	72	12
14	1	73	1
Total	73		

Example 3: Consider the marks (out of 100) of 50 students as below :

40,	39,	43,	62,	30,	47,	33,	31,	17,	28
36,	29,	40,	32,	39,	24,	57,	42,	15,	30
50,	52,	47,	65,	31,	07,	37,	47,	17,	20
25,	53,	65,	85,	89,	56,	55,	41,	43,	10
44,	40,	69,	22,	40,	65,	39,	36,	71,	12

Prepare grouped frequency distribution.

Solution:

Table

Marks	Tally Marks	Frequency
0 - 34	卌 卌 卌 |	16
35 - 44	卌 卌 卌 |||	18
45 - 59	卌 ||||	9
60 - 100	卌 ||	7
Total		50

5.3 PRESENTATION OF DATA

Presentation of data is of great importance now a days because data should be in such a manner that pleasing to our eyes and grab our attention. Presentation of data refers to an exhibition or putting up data in an attractive and useful manner such that it can be easily interpreted. The three main forms of presentation of data are:

1. **Textual presentation**

In such form of presentation, data is simply mentioned as normal text i.e. generally in a paragraph. This is commonly used when the data is not very large. This kind of representation is useful when we are looking to supplement qualitative statements with some data. It just has to be a statement that serves as a fitting evidence to our qualitative evidence and helps the reader to get an idea of the scale of a phenomenon.

For example, “the 2002 earthquake proved to be a mass murderer of humans. As many as 10,000 citizens have been reported dead”. The textual representation of data simply require some intensive reading. This is because the quantitative statement just serves as an evidence of the qualitative statements and one has to go through the entire text before concluding anything. Further, if the data under consideration is large then the text matter increases substantially. As a result, the reading process becomes more intensive, time-consuming and cumbersome.

2. **Data Tables/Tabular Presentation**

A table is a structure which facilitates representation of large amounts of data in an attractive manner so that these can be easily understood and organized. Here, the data are organized in a set of rows and columns. This is one of the most widely used form of presentation of data since data tables are easy to construct and read.

3. **Diagrammatic presentation**

As the name suggest data can be presented through different types of diagrams. One can easily understand some of the diagrams only through a glance and some diagrams require special attention.

Here, we will be studying only tabular presentation, i.e. data tables in some detail. Further, diagrammatic presentation will be studied in detail in next chapter.

5.3.1 TABULAR PRESENTATION OF DATA

A table is a structure which facilitates representation of large amounts of data in an attractive manner so that these can be easily understood and organized. Here, the data are organized in a set of rows and columns. This is one of the most widely used form of presentation of data since data tables are easy to construct and read.

I. Components of Data Tables

(i) **Table Number**: A table should have a specific number for easy access and identification. This number can be used for reference and leads us directly to the data mentioned in that particular table.

(ii) **Title:** A table without title is a waste and of no use because a title is an identity to a table. There should be a title which tells us about the data it contains, time period of study, place of study and the nature of classification of data.

(iii) **Headnotes:** As we know that title of a table contains most of the information required by us but there is lack of full information about data contained in table. Thus, headnotes displays more information about the table.

(iv) **Stubs:** Stubs are titles of the rows in a table which gives information about that row.

(v) **Caption:** Caption are opposite to the stubs which means caption is the title of a column in the data table. It indicates the information contained in a column.

(vi) **Body or field:** The body of a table is the content of a table and each item in a body is known as a 'cell'.

(vii) **Footnotes:** Footnotes are rarely used. In effect, they supplement the title of a table if required.

II. The Advantages of Tabular Presentation

(a) It is easy to represent a large amount of data in a table.

(b) It is very easy to use statistical tools on a data table.

(c) It is easy to compare data in the form of Table. In a data table, the rows and columns which are required to be compared can be placed next to each other.

(d) It is very easy to construct a data table and present the data in a manner so that these can be easily understandable by us.

(e) A data table is helpful to save time and space.

(f) Diagrammatic presentation becomes easy through table data.

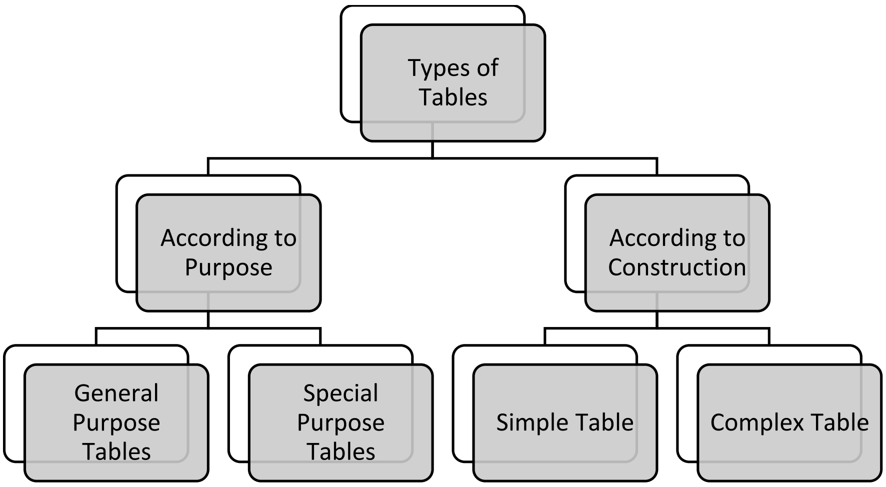
III. General Rules for Constructing a Good Table

It is very easy to construct a table but construction of a good table depends on the objectives of the study. Further, construction of a good table requires an intelligent statistician. There is no hard and fast rule to construct a good table but still there are some necessities to be kept in mind. Some points are discussed as under:

(a) Title
(b) Specific size of table
(c) Special emphasis on certain items such as headings of raw and columns
(d) Headings should be written in the singular form
(e) Abbreviations should not be used
(f) Footnotes should be given only if required.
(g) The units used in the table must be defined above columns
(h) Sub-totals and grand total must be present in a table
(i) Percentages figures should be clearly shown in the table
(j) Extent of approximation should be defined clearly
(k) Source of data should be given in the footnotes of the table
(l) Table must be simple, attractive and economical in space

IV. Types of Table

There are various types of tables but these can be combined according to some basis i.e. According to Purpose and According to Construction.



I. According to Purpose

You may construct a table according to your purpose which may be general or specific.

General Purpose Tables

General purpose tables also known as the reference tables, provide information for general use or reference. These tables are simply data bank for researchers for their various studies. For example: Census Report of India.

Special Purpose Tables

Special purpose tables, also known as summary or analytical tables, provide information for particular discussion. These are simple and small tables which are limited to the problem.

II. According to Construction

These can be segregated into two parts which are explained below:

Simple Table

As the name suggest these are very simple table with a single variable and also known as one way table. In a simple table only one characteristic is shown.

For example:

The marks secured by a batch of students in a class test are displayed in Table 3.8

Marks of Students

Marks	0 - 10	10 - 20	20 - 30	30 – 40	40 - 50	50 - 60
Number of Students	10	12	17	20	15	6

Complex Table

In a complex table, two or more characteristics are shown. These may be of two types i.e. Two-way table and Higher order table.

Two-way Table:- Such a table shows two characteristics and is formed when either the stub or the caption is divided into two coordinates parts. For example:

Marks of Students

Marks	Number of Students		Total
	Males	Females	
30 – 40	8	6	14
40 – 50	16	10	26
50 – 60	14	16	30
60 - 70	12	8	20
70 – 80	6	4	10
Total	56	44	100

Higher Order Table or Manifold Table:- When three or more characteristics are represented in the same table, such a table is called higher order table. For example:

Marks of Students

Marks	Males		Total	Females		Total	Total		Total
	Urban	Rural		Urban	Rural		Urban	Rural	
30 – 40	4	4	8	4	2	6	8	6	14
40 – 50	10	6	16	5	5	10	15	11	26
50 – 60	8	6	14	9	7	16	17	13	30
60 - 70	7	5	12	5	3	8	12	8	20
70 – 80	5	1	6	2	2	4	7	3	10
Total	34	22	56	25	19	44	59	44	100

5.4 CHECK YOUR PROGRESS

1 The frequency distribution of two variable is known as ________________.

2 The upper class limit _____________ but the lower class limits include in the Exclusive Method.

3 Both the upper and the lower class limits are _________ in the Inclusive Method.

4 In ____________ presentation, data is simply mentioned as normal text i.e. generally in a paragraph.

5 In a _________ table only one characteristic is shown.

5.5 SUMMARY

Data handling and interpretation is an important topic under Statistics as well as Secondary Mathematics that impacts our daily lives. Data representation is helpful for students to use its techniques for various projects. Groups will learn a great deal from presenting their work to the class as a whole. Once you know the techniques of classification then it will become easy to construct a frequency distribution in both types i.e. continuous and discrete variable. When data handing become easy then you will be able to use these data according to your purpose. Thus, it is very important to be aware about the techniques of presentation and forms of presentation of data. Further, tabular presentation is one of the easiest form of presentation and everyone can easily understand data in the form of a table. Diagrammatic presentation is also important form of presentation but it is not so easy to understand diagrammatic presentation because some of diagrams are easy to understand and some of them are very difficult to understand. Thus, data presentation helps a lot in our daily life as well as in business life.

5.6 KEYWORDS

1. **Univariate Distribution:** The frequency distribution of a single variable is called a Univariate Distribution.
2. **Classification:** Classification is arranging or organising similar things into groups or classes.
3. **Geographical Classification:** When data are classified with reference to geographical locations such as countries, states, cities, districts, etc. it is known as Geographical Classification. It is also known as ‘Spatial Classification’.
4. **Individual Series:** Individual Series is a statistical series in which the all the observations are listed out and these observations have a single frequency.

5. **Title:** A table without title is a waste and of no use because a title is an identity to a table. There should be a title which tells us about the data it contains, time period of study, place of study and the nature of classification of data.

5.7 SELF-ASSESSMENT TEST

Q.1 What do you mean by Classification? Explain the purpose of classification in data handling.

Q.2 Do you think classification of data is advantageous in data handling? Explain with an example from your daily life.

Q.3 What do you mean by frequency distribution? Explain different type of frequency distribution in detail.

Q.4 Differentiate Discrete and Continuous frequency distribution.

Q.5 Explain the guidelines for construction of a good table.

Q.6 Use the data in Table 3.2 that relate to monthly household expenditure (in Rupee) on food of 50 households and

TABLE I

Monthly Household Expenditure (in Rupees) on Food of 50 Households

1904	1559	3473	1735	2760
2041	1612	1753	1855	4439
5090	1085	1823	2346	1523
1211	1360	1110	2152	1183
1218	1315	1105	2628	2712
4248	1812	1264	1183	1171
1007	1180	1953	1137	2048
2025	1583	1324	2621	3676
1397	1832	1962	2177	2575
1293	1365	1146	3222	1396

- Obtain the range of monthly household expenditure on food.
- Divide the range into appropriate number of class intervals and obtain the frequency distribution of expenditure.
- Find the number of households whose monthly expenditure on food is

(a) Less than Rupee 2000
(b) More than Rupee 3000
(c) Between Rupee 1500 and Rupee 2500

Q.7 In a city 45 families were surveyed for the number of domestic appliances they used. Prepare a frequency array based on their replies as recorded below:

1	3	2	2	2	2	1	2	1	2	2	3	3	3	3
3	3	2	3	2	2	6	1	6	2	1	5	1	5	3
Q 4		2	7	4	2	4	3	4	2	0	3	1	4	3

Q.8 Prepare a frequency distribution by inclusive method taking class interval of 7 from the following data:

28	17	15	22	29	21	23	27	18	12	7	2	9	4	6
1	8	3	10	5	20	16	12	8	4	33	27	21	15	9
3	36	27	18	9	2	4	6	32	31	29	18	14	13	19
15	11	9	7	1	5	37	32	28	26	24	20	19	25	20

5.8 ANSWERS TO CHECK YOUR PROGRESS

1. Bivariate Analysis
2. Excluded
3. Included
4. Textual Presentation
5. Simple Table

5.9 REFERENCES/SUGGESTED READINGS

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Business Mathematics-II	
Lesson No: 6	**Author: Dr. Vizender Singh**
Data Representation and Interpretation II: Diagrammatic and Graphic Presentation with Benefits and Limitation	**Vetter: Prof. Kuldeep Bansal**

Lesson Structure

6.0 LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Presentation of data using diagrams, their benefits and limitation
- Different types of diagrams
- Presentation of data using graphs, their benefits and limitation
- Different type of graphs

6.1 INTRODUCTION

In the previous chapter we have seen how to condense the mass data by the method of classification and tabulation. It is not so easy to understand figures every time and these may not be interesting for everyone. Further, some of figures are very confusing and complicated which creates problems while analyzing. One of the most convincing and appealing ways in which statistical results may be represented is through graphs and diagrams. Diagrams can be more easily compared, and can be interpreted by a layman. Diagrams are more attractive and have a visual appeal. This is the reason that diagrams are often used by businessmen, newspapers, magazines, journals, government agencies and also for advertising and educating people. In this chapter we will discuss about the diagrammatic presentation of data and graphical presentation of data.

William play fair - 1759 - 1823 said in his 'Commercial and political Atlas of 1801' that "Information take days through data presented in tables, but can be obtained in minutes through diagrams."

Cold figures are uninspiring to most people. Diagrams help us to see the pattern and shape of any complex idea, just as a map gives a bird's eye view of wide sketch of a country, so diagrams help us to visualize the whole meaning of a numerical complex at single glance. Diagrams register a meaningful impression almost before you think ...." (Moroney).

"The important point that must be borne in mind at all times that the pictorial representation chosen for any situation must depict the true relationship and point out the proper conclusion. Above all the chart must be honest." .... C. W. LOWE.

Data can be visually presented as shown in the Fig.1, i.e. Diagrammatic and Graphic. These are also known as visualization because they create an image in the mind of the reader. Both of these representation of data further categorized among different parts. This classification is discussed in the next section of this chapter.

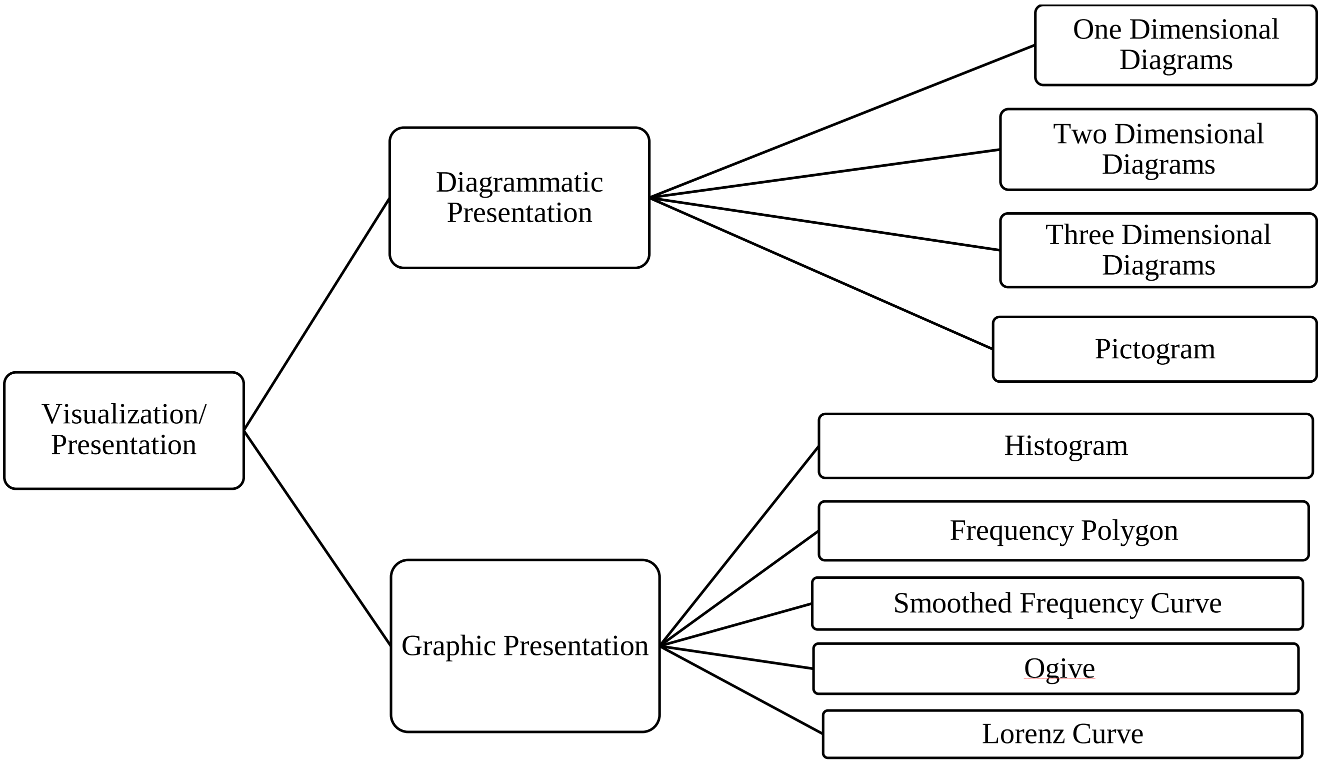


Fig.1 Types of Data Presentation/Visualization

These are discussed one by one in this chapter. First of all we will discuss about Diagrammatic Presentation of data and later on we will focus on Graphical Presentation of data.

6.2 DIAGRAMMATIC PRESENTATION OF DATA

Diagrammatic Presentation of Data provides an immediate understanding of the real situation. Diagrammatic presentation of data converts the highly complex ideas included in numbers into more concrete and quickly understandable form. Diagrams may be less certain but are much more efficient than tables in displaying the data. There are many kinds of diagrams in general use such as Pictograms, Cartograms, Bar Diagrams & Pie Diagrams etc. Diagrams help in visual comparison and have a bird's eye view. Diagrams are different geometrical shape such as bars, circles etc. Diagrams are based on scale but are not confined to points or lines. They are more attractive and easier to understand than tables and graphs.

6.2.1 BENEFITS OF DIAGRAMMATIC PRESENTATION OF DATA

1. Diagrammatic presentation is very attractive and easy to understand.
2. There is no need of technical knowledge and expertise to form and understand diagrams.
3. Diagrams require less time and efforts as compared to graphs.
4. Visual things are more memorisable and had a lasting impression.
5. Diagrams can be understood by everyone because there is no language barrier.
6. Diagrams are helpful to represent huge data in a simplified and intelligible form.
7. Diagrams are helpful in making comparison of data.
8. Diagrams also provides hidden information about the data.

6.2.2 GENERAL RULES FOR CONSTRUCTING DIAGRAMS

1. The diagrams should be simple and clear.
2. There should be a suitable title without damaging clarity.
3. You should keep in mind the height and width must be maintained forming a diagram.
4. You should select a proper scale such as it may be in even numbers or multiples of five or ten.
5. There should be footnotes under the diagram.

6. An index should be inserted for explaining different lines, shades and colours.

6.2.3 TYPES OF DIAGRAMS

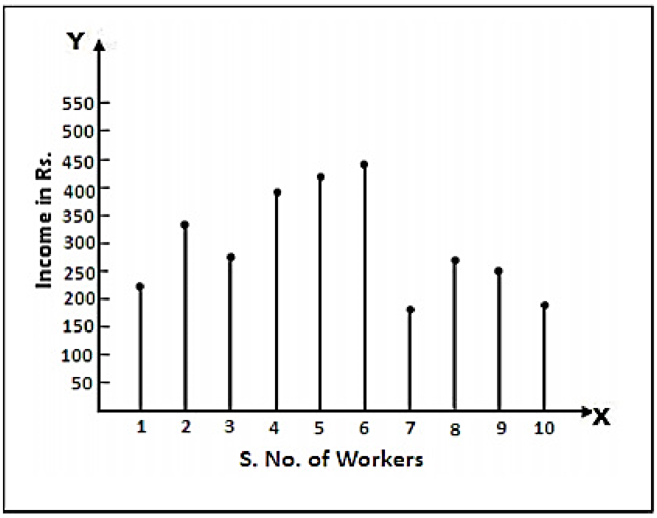
As we have discussed earlier in the figure 1, diagrammatic presentation can be divided into four parts which are explained below:

I. One Dimensional Diagrams

One dimensional diagrams are the diagrams in which only one dimension is included i.e. height. Here, width or thickness is not measured. These diagrams may be seen in the form of lines or bars. These diagrams are divides into five parts discussed as under:

Line Diagram refers to the diagram where you have to present many items and there is not much difference in their values. These are simply formed through using a vertical or horizontal line for each item according to scale and the space among lines is kept uniform. This is the easiest method of data representation which makes comparison easy but it is not an attractive method. It can be shown as diagram I:

Here, Income of 10 workers is shown according to their series number. Income of worker is shown on Y axis and Series number of workers is shown on X axis.

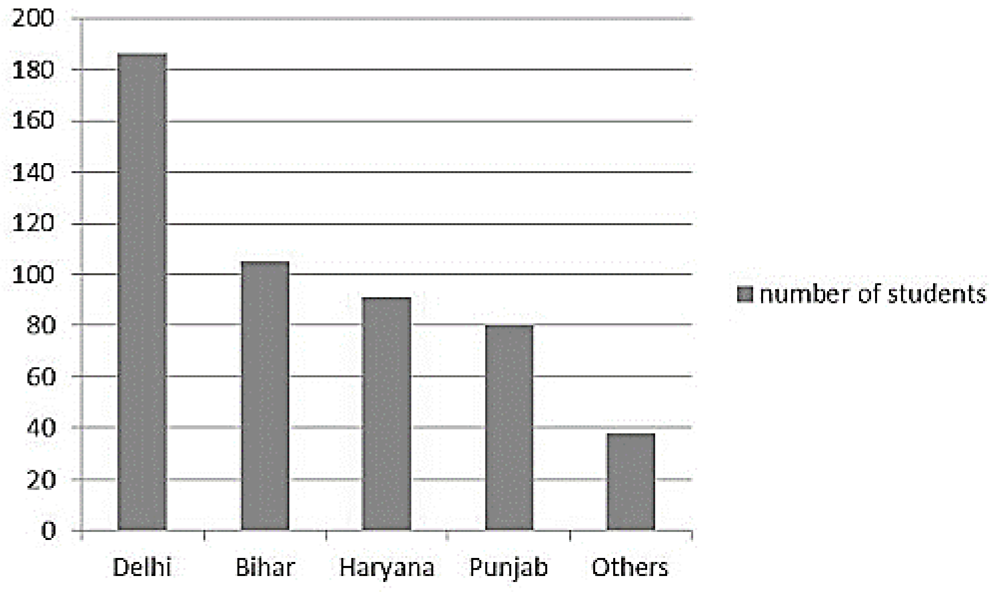


Line Diagram of Income of 10 workers

Diagram I

Simple Bar Diagram represents only one variable. For example sales, production, population figures etc. for various years may be shown by simple bar charts. Since these are of the same width and vary only in heights or lengths. Further, it becomes very easy for readers to study the relationship. Simple bar diagrams are very popular in practice. A bar chart can be either vertical or horizontal.

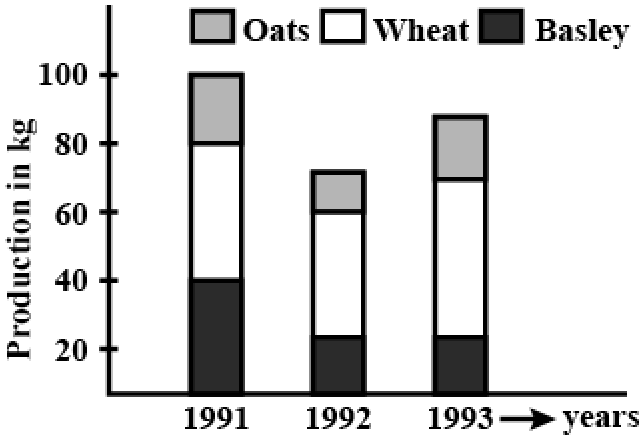
For example, Diagrammatic representation of data relating to number of students in different states is given below:



Simple Bar Diagram II

Sub - divided Bar Diagram: While constructing such a diagram, the various components in each bar should be kept in the same order. A common and helpful arrangement is that of presenting each bar in the order of magnitude with the largest component at the bottom and the smallest at the top. The components are shown with different shades or colors with a proper index.

For example, Sub-divided bar diagram is shown production of Oats, Wheat and Basley in different years.



Sub-divided Bar Diagram III

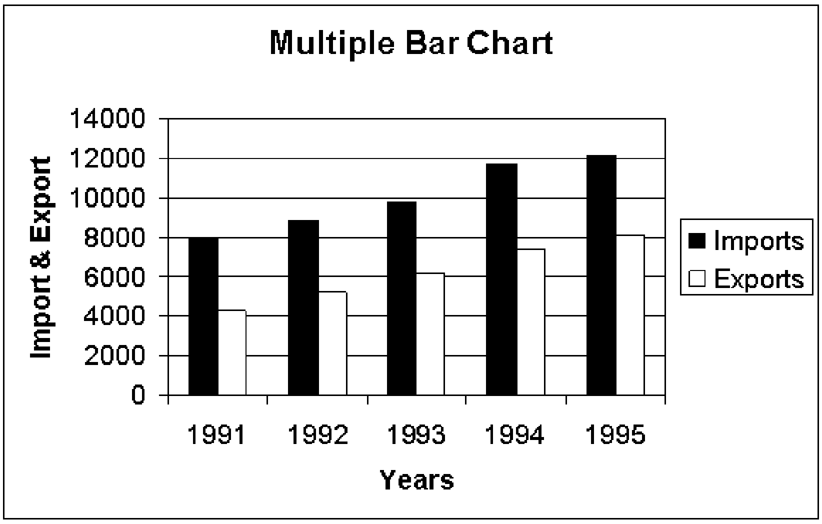
Multiple Bar Diagram: This method can be used for data which is made up of two or more components. In this method the components are shown as separate adjoining bars. The height of each bar represents the actual value of the component. The components are shown by different shades or colors. Where changes in actual values of component figures only are required, multiple bar charts are used.

For example, Draw a multiple bar chart to represent the imports and exports of Canada (values in $) for the years 1991 to 1995.

Years	**Imports**	**Exports**
1991	7930	4260
1992	8850	5225
1993	9780	6150
1994	11720	7340
1995	12150	8145

Table I: Import and Export data of Canada from 1991 to 1995

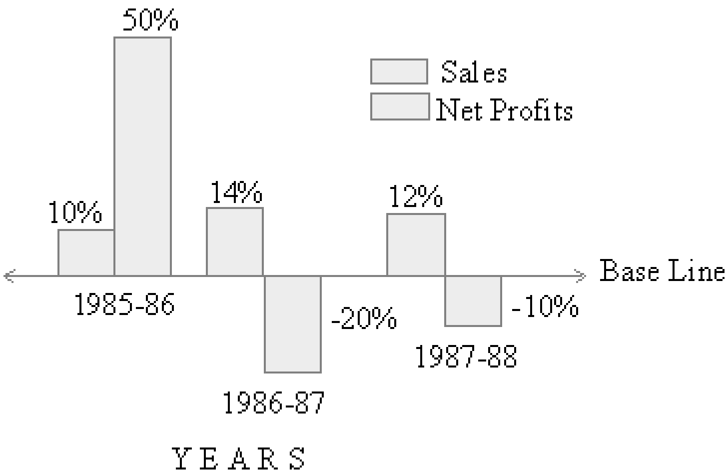
Sol:



Multiple Bar Diagram IV

Deviation Bar Diagram: Deviation bars are used to represent net quantities - excess or deficit i.e. net profit, net loss, net exports or imports, swings in voting etc. Such bars have both positive and negative values. Positive values lie above the base line and negative values lie below it.

For example, Sales and Net Profit are shown in the Deviation Bar Diagram.



Deviation Bar Diagram V

All the above diagram represents one dimensional Diagrams. In the next part, we will discuss two dimensional Diagrams.

II. Two Dimensional Diagrams

Two dimensional diagrams are those diagrams where both the dimension length as well as width of the bar are considered for construction of diagrams. These diagrams are also known as "Area" or "Surface" diagrams. There are three types of area diagrams such as Rectangles, Squares and Pie Diagrams.

Rectangles are diagrams which are used to represent the magnitude of two or more values and rectangles are placed side by side so that comparison can be made. These diagrams represents two different characteristics of data. We may represent data as they are given or these can be converted into percent and then subdivided into parts according to the length.

For Example,

Represent the following data by sub-divided percentage rectangular diagram.

Items of Expenditure	Family A (Income Rs.5000)	Family B (income Rs.8000)
Food	2000	2500
Clothing	1000	2000
House Rent	800	1000
Fuel and lighting	400	500
Miscellaneous	800	2000
Total	5000	8000

Table II: Items of expenditure with Income of Family A & B

Sol: First of all, these data are converted into percentage and then subdivided into different section of rectangular diagram.

Items of Expenditure	Family A		Family B	
	Rs.	Y	Rs.	Y
Food	2000	40	2500	31
Clothing	1000	20	2000	25
House Rent	800	16	1000	13
Fuel and Lighting	400	8	500	6
Miscellaneous	800	16	2000	25
Total	5000	100	8000	100

Table III: Items of expenditure with Income of Family A & B

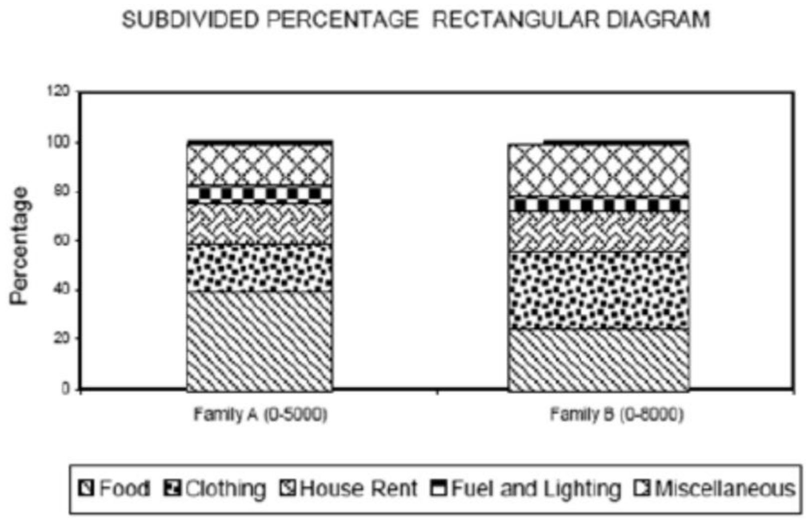


Diagram VI: Subdivided Percentage Rectangular Diagram

Square diagrams requires square root of the given data which provides the measurement of the sides of the square. For example, Yield of rice in Kgs. per acre of five countries are:

Country	U.S.A	Australia	U.K	Canada	India
Yield of rice in Kgs per acre	6400	1600	2500	3600	4900

Table IV: Yield of Rice in Kgs Per Acre of Different Countries

Represent the above data by a square diagram.
Sol: First of all we calculate the square root of data as follows:

Country	Yield	Square root	Side of the square in cm
U.S.A	6400	80	4
Australia	1600	40	2
U.K.	2500	50	2.5
Canada	3600	60	3
India	4900	70	3.5

Table V: Yield of Rice in Kgs Per Acre of Different Countries

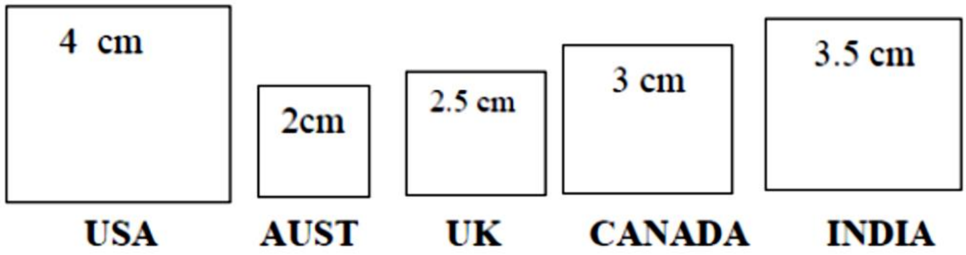


Diagram VII: Square Diagram Representing Yield of Rice in Kgs Per Acre of Different Countries

Pie Charts or Diagram consists of a circle in which the radii divide the area into sectors. Further, these sectors are proportional to the values of the component items under investigation. Also, the whole circle represents the entire data under investigation.

Steps to draw a Pie Chart

- Express the different components of the given data in percentages of the whole
- Multiply each percentage component with 3.6 (since the total angle of a circle at the center is 360°)
- Draw a circle
- Divide the circle into different sectors with the central angles of each component
- Shade each sector differently

Let us take **an example**, consider the yearly expenditure of a Mr. Ted, a college undergraduate.

Tuition fees	$ 6000
Books and lab.	$ 2000
Clothes / cleaning	$ 2000
Room and boarding	$ 12000
Transportation	$ 3000
Insurance	$ 1000
Sundry expenses	$ 4000

Total expenditure	$ 30000

Table VI: Yearly Expenditure

Now as explained above, we calculate the angles corresponding to various items (components).

Tuition fees $= \frac{6000}{30000} \times 360^0 = 72^0$

Book and lab $= \frac{2000}{30000} \times 360^0 = 24^0$

Clothes / cleaning $= \frac{2000}{30000} \times 360^0 = 24^0$

Room and boarding $= \frac{12000}{30000} \times 360^0 = 144^0$

Transportation $= \frac{3000}{30000} \times 360^0 = 36^0$

Insurance $= \frac{1000}{30000} \times 360^0 = 12^0$

Sundry expenses $= \frac{4000}{30000} \times 360^0 = 48^0$

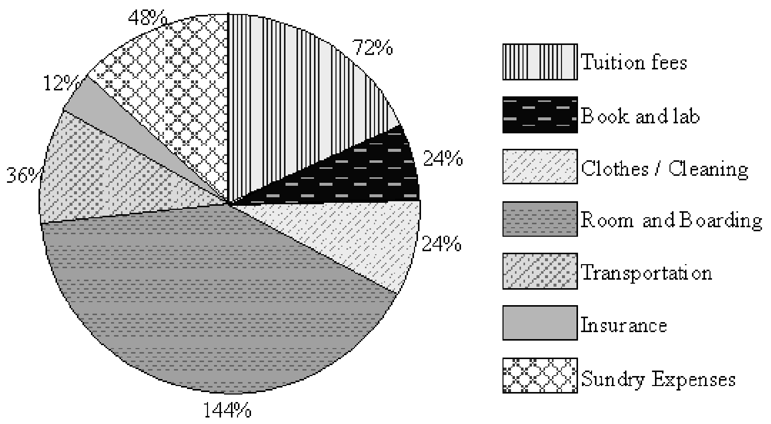


Diagram VIII : Pie Chart of total expenditure

III. Three Dimensional Diagrams

Three dimensional diagrams are the diagrams in which three dimension are taken into account. These dimensions are length, width and height. These diagrams are also known as Cubic Diagram and these diagrams may be drawn in the form of cylinders, blocks, spheres, etc.

IV. Pictogram

As the name suggested that pictogram uses appropriate pictures to represent data. It is also known as Picture Graph or Pictograph. These are very attractive and create a lasting effect on viewer mind.

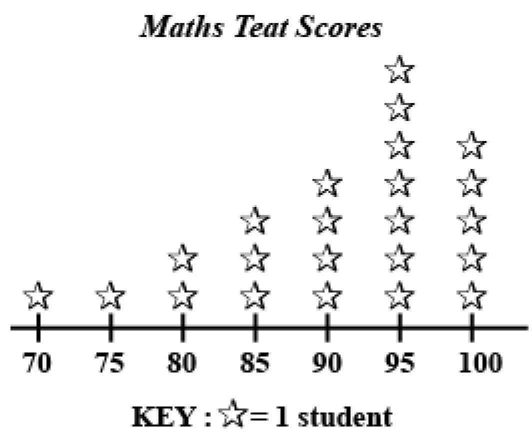


Diagram IX : Pictogram representing Math Test Score

6.2.4 LIMITATIONS OF DIAGRAMMATIC PRESENTATION OF DATA

As we studied that the above diagrams are very attractive as well as very easy to form. But, still there are some deficiencies which are explained below:

1. These diagrams do not provide detailed information.

2. Diagrams can be easily misinterpreted.

3. Diagrams can take much time and labour.

4. Exact measurement is not possible in diagrams.

6.3 GRAPHIC PRESENTATION OF DATA

A graph is a visual representation of data by a continuous curve on a graph paper. Graphs are more attractive than a table or figure and revealing their inner pattern. A common man can even understand the message given in the graph. Graphical Representation is a way of analysing numerical data. It exhibits the relation between data, ideas, information and concepts in a diagram. Graphs enable us in studying the cause and effect relationship between two variables. Graphs help to measure the extent of change in one variable when another variable changes by a certain amount. It is easy to understand and it is one of the most important learning strategies. It always depends on the type of information in a particular domain. There are different types of graphical representation. There are some important forms of graphs i.e. Histogram, Frequency Polygon, Smooth Frequency Curve, Ogive and Lorenz Curve. These are discussed in detail in the next section i.e. Type of Graphs.

6.3.1 BENEFITS OF GRAPHIC PRESENTATION OF DATA

1. Graphs represent complex data in a simple form.

2. Values of median, mode can be found through graphs.

3. Graphs create long lasting effect on people’s mind.

4. Graphs are attractive and impressive.

5. Graphs make data simple and intelligible.

6. Graphs make comparison possible.

7. Graphs save time and labour.

8. Graphs have universal utility.

9. Graphs give more information.

6.3.2 GENERAL RULES FOR CONSTRUCTING A GRAPH

1. First of all, one should focus on the title of a graph which should be simple and clear.

2. Then it is important to mention the measurement unit in the graph so that values can determined easily.

3. It is necessary to choose a proper scale so that we can present data in an accurate manner.

4. It is important to provide different colours, shades, designs in the graph for better understanding.

5. In the bottom of graph, source of information must be given.

6. Graph should be simple and clear without any ambiguity.

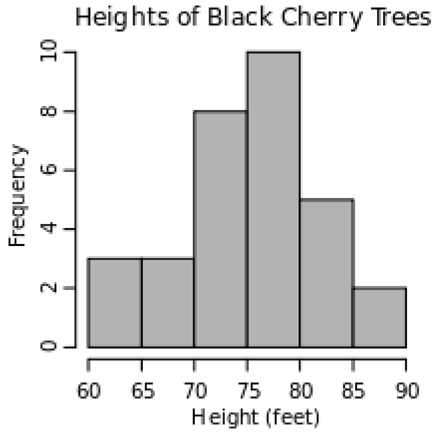
7. Graph is a form of visualisation aid for presentation of data so, one must choose correct size, letters and colours.

6.3.3 TYPES OF GRAPHS

There are major five types of graphs which are explained below with the help of an example:

I. Histogram

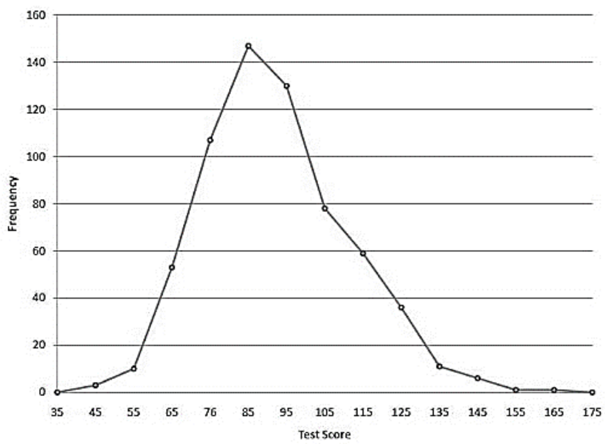
A histogram is a graph showing the frequency of occurrence of each value of the variable being analysed. In histogram, data are plotted as a series of rectangles. Class intervals are shown on the ‘X-axis’ and the frequencies on the ‘Y-axis’if the classes are of equal width and frequency density (f/c) on 'Y-axis' if the classes are of un equal width. The height of each rectangle represents the frequency or frequency density of the class interval. Each rectangle is formed with the other so as to give a continuous picture. Such a graph is also called staircase or block diagram. However, we cannot construct a histogram for distribution with open-end classes.



Histogram

II. Frequency Polygon

If we mark the midpoints of the top horizontal sides of the rectangles in a histogram and join them by a straight line, the figure so formed is called a Frequency Polygon. This is done under the assumption that the frequencies in a class interval are evenly distributed throughout the class. The area of the polygon is equal to the area of the histogram, because the area left outside is just equal to the area included in it. Another method of drawing frequency polygon is on the X axis draw the mid points and on the Y axis the frequency density (f/c) join the points by straight line to obtain frequency polygon.



Frequency Polygon

III. Smoothed Frequency Curve

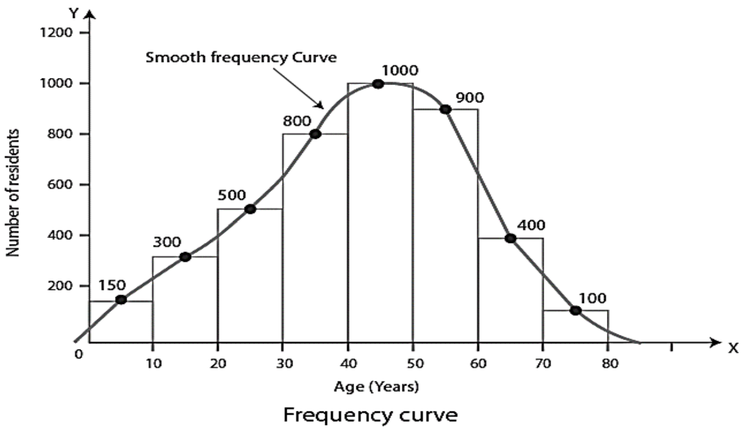
If the middle point of the upper boundaries of the rectangles of a histogram is corrected by a smooth freehand curve, then that diagram is called frequency curve. The curve should begin and end at the base line.

For example,

Make a frequency curve of the following data.

Age (Years)	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No of Residents	150	300	500	800	1,000	900	400	100

Sol: The given data set is first converted into a histogram. Mid-points o the top of the rectangles of the histogram are marked. These points are joined through a freehand smoothed curve, as given below.



Smooth Frequency Curve

IV. Ogive

The cumulative frequency gives the cumulative frequency of each of the class. The curve table is obtained by plotting cumulative frequencies is called a cumulative frequency curve or an ogive. There are two type of ogive namely:

1. The ‘less than ogive’

   In less than ogive method we start with the upper limits of the classes and go adding the frequencies. When these frequencies are plotted, we get a rising curve.

2. The ‘more than ogive’

   In more than ogive method, we start with the lower limits of the classes and from the total frequencies we subtract the frequency of each class. When these frequencies are plotted we get a declining curve.

For example,

Draw the ‘less than’ and ‘more than’ ogive on the same graph paper from the data given below:

Weekly wages (₹)	No of workers
0-20	10
20-40	20
40-60	40
60-80	20
80-100	10

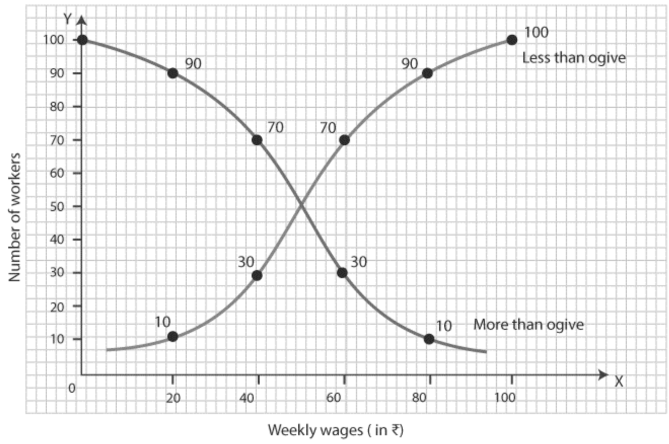
Solution: (i) ‘Less than’ method

Weekly wages (₹)	C.F
Less 20	10
Less 40	30
Less 60	70
Less 80	90
Less 100	100

(ii) ‘More than’ method

Weekly wages (₹)	C.F
Less 20	10
Less 40	30
Less 60	70
Less 80	90
Less 100	100

Both ‘less than’ and ‘more than’ ogives based on the above data are presented in the following graph.



‘Less than’ and ‘More than’ Ogives

V. Lorenz Curve

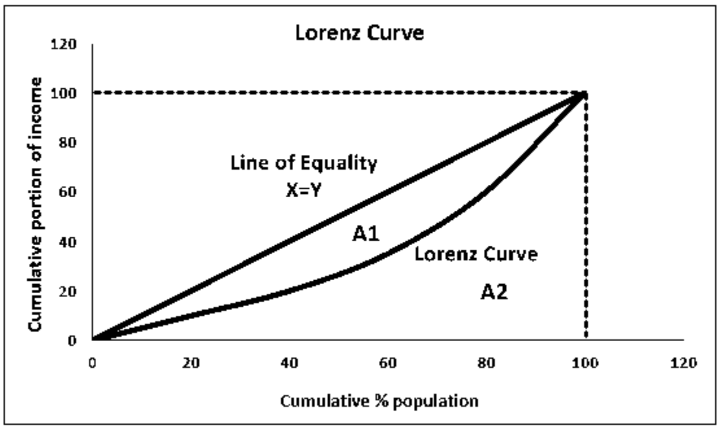
It is a graphical method of studying dispersion among data and this curve was introduced by Max. O. Lorenz, a great Economist as well as Statistician. It is a percentage of cumulative values of one variable in combined with the percentage of cumulative values in other variable and then Lorenz curve is drawn. This curve starts with the origin (0,0) and ends at (100,100). For example, Let us consider an economy with the following population and income statistics:

Population %	Income Portion %
0	0
20	10
40	20
60	35
80	60
100	100

And for the line of perfect equality, let us consider this table:

Population %	Income Portion %
0	0
20	20
40	40
80	80
100	100

Let us now see how a graph for this data actually looks:



6.3.4 LIMITATIONS OF GRAPHIC PRESENTATION

As we discussed that graphical presentation is an eye catching way to represent data but one should be very careful for constructing it. There are various limitations of graphical presentation of data i.e.

1. Sometimes graphs only provides half information to a common man and deep understanding of graphs require experts.
2. Graphs only provides limited information.
3. Graphical presentation of data is very costly because it involves images, colours and paints.
4. Graphical presentation requires a lot of time and energy.
5. There is high possibilities of error and mistake in the graphical presentation of data.
6. Graphs presents full information of data which cause lack of secrecy.
7. It is very difficult to select a suitable method for graphical presentation of data.

6.4 CHECK YOUR PROGRESS

1. Why Diagrammatic Presentation is better than Tabulation of Data?

2. Which Bar Diagram is used to show two or more characteristics of the Data?

3. Mention the total of the degrees of all the angles formed at the centre of a Circle.

4. A histogram is a graphical representation of ___________.

5. A sector diagram is also known as __________.

6.5 SUMMARY

Now, you are able to learn how collected data can be presented through various forms of presentations or visualization i.e. Diagrammatic and Graphical presentations. Moreover, at this point you are able to understand the different forms of diagrams and graphs. You can easily construct diagrams and graphs, and represent data according to your use. You have an idea about the rules of constructions for presentation of data and you are also aware about the merits and demerits of using the presentation method. Thus, you can make presentation of data meaningful,

Comprehensive and purposeful. Further, presentation of data is very useful for every section of society i.e. whether you are a student, a businessmen or a common make. Thus, through understanding presentation of data you will be able to present data in an efficient manner, you can use appropriate method of presentation and easily spread information to others.

6.6 KEYWORDS

Presentation of Data refers to the organization of data into tables, graphs or charts, so that logical and statistical conclusion can be derived from the collected measurement.

A bar chart consists of a set of bars whose heights are proportional to the frequencies that they represent.

Histogram is a set of vertical bars whose areas are proportional to the frequencies of the classes that they represent.

Ogive is a cumulative frequency graphs drawn on natural scale to determine the values of certain factors like median, Quartile, Percentile etc.

Three dimensional diagrams are the diagrams in which three dimension are taken into account. These dimensions are length, width and height. These diagrams are also known as Cubic Diagram and these diagrams may be drawn in the form of cylinders, blocks, spheres, etc.

6.7 SELF-ASSESSMENT TEST

1. Represent the following data in a bar chart.

The amount (in thousands of litres) of petrol sold at a petrol station during a month was

Type of petrol	Leaded	Unleaded	Diesel
Number of litres (x 1000)	45	35	20

2. Use the following raw data of the length (mm) of nails found in packets of ‘assorted nails’.

11	48	53	32	28	15	17	45	37	41
55	31	23	36	42	27	19	16	46	39
41	28	43	36	21	51	37	44	33	40
15	38	54	16	46	47	20	18	48	29
31	41	53	18	24	25	20	44	13	45

a) Make a grouped frequency table taking class intervals 10 -14, 15 - 19, etc., and draw a histogram.

b) Make a grouped frequency table taking class intervals 10 - 19, 20 -29, etc., and draw the histogram.

Compare the two representations of the data.

3. Represent the following data by sub-divided percentage rectangular diagram.

Items of Expenditure	Family A		Family B	
	Rs.	Y	Rs.	Y
Food	2000	40	2500	31
Clothing	1000	20	2000	25
House Rent	800	16	1000	13
Fuel and Lighting	400	8	500	6
Miscellaneous	800	16	2000	25
Total	5000	100	8000	100

4. What do you mean by Diagrammatic presentation of data? Explain the major rule for constructing a diagram.
5. What do you understand by Graphical presentation of data? How it is different from Diagrammatic presentation of data?
6. Comparing the two representations, pie chart and pictogram, list some advantages and disadvantages of each.
7. The number of teacher trained for the Senior Secondary School in Botswana are tabulated:

Year	1996	1997	1998	1999
Number trained	81	105	115	184

a) Represent this data in a line graph and comment on the trend.

b) If the trend is continuous what number of Senior Secondary school teachers do you expect to be trained in the year 2000?

8. Display the following data in a pie chart and pictogram.

The type of vehicles coming to a petrol station during one day are tabulated below:

Person cars 26 Lorries 12 Busses 8 Combis 14

9. Get information from your school office about the CBSE result (2019) for the students of Class XII in your school. Draw a bar diagram (showing their aggregate marks classified as 1st division, 2nd division and 3rd division).

10. Here is an exercise for the students of Class XI. Draw a programme to conduct direct personal oral investigation of all the students of your school. Find out which mode of transport they use to come to the school. Present your information in terms of a pie diagram.

6.8 ANSWERS TO CHECK YOUR PROGRESS

1. It makes data more attractive as compared to tabulation and helps in visual comparison.
2. Multiple Bar Diagram
3. 360°
4. A Frequency Distribution
5. Pie Diagram

6.9 REFERENCES/SUGGESTED READINGS

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